Categories of Digraphs and Automata

Fariba Karimi and Chrystopher Nehaniv

Woflson / Royal Society Biocomputation Laboratory
Science and Technology Research Institute
University of Hertfordshire, U.K.

2st BIOMICS Workshop
St Andrews, 19 June 2014

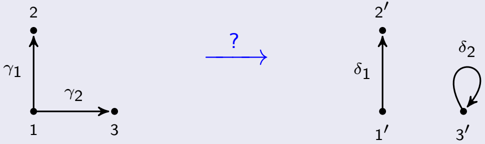
Labelled Digraphs

The category of labelled digraphs consists of all directed graphs with labelled edges,
$\Gamma = (V, E, \lambda \colon E \to \Lambda)$ where $E \subseteq V \times V$, Λ is the set of labels, and λ assigns a label to each edge. A morphism $\varphi \colon \Gamma \to \Gamma'$ in the category of digraphs is defined as a 3-tuple
$\varphi = (\varphi^{Vertex} \colon V \to V',\ \varphi^{Edge} \colon E \to E',\ \varphi^{Label} \colon \Lambda \to \Lambda')$, such that always

$$\varphi^{Edge}(v_1, v_2) = (\varphi^{Vertex}(v_1), \varphi^{Vertex}(v_2)), \quad \forall v_1, v_2 \in V, \tag{1}$$

and $$\lambda'(\varphi^{Edge}(e)) = \varphi^{Label}(\lambda(e)), \qquad \forall e \in E. \tag{2}$$

- The initial object is the empty labelled digraph with no vertices, no edges, and empty label set.
- A terminal labelled digraph has exactly one vertex, one edge and one label.


2
γ_1
γ_2
1
3
?
$2'$
δ_1
δ_2
$1'$
$3'$

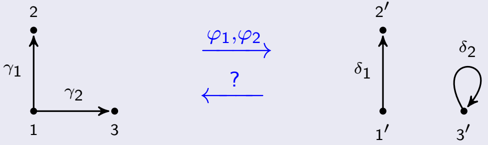

2
1
3
γ1
γ2
φ1,φ2
2′
1′
3′
δ1
δ2

- $\varphi_1 = (\varphi_1^{Vertex}: , \varphi_1^{Edge}, \varphi_1^{Label})$, where

$\varphi_1^{Vertex}:$	$\varphi_1^{Edge}:$	$\varphi_1^{Label}:$
$1 \mapsto 1'$	$(1,2),(1,3) \mapsto (1',2')$	$\gamma_1, \gamma_2 \mapsto \delta_1$
$2,3 \mapsto 2'$		

- $\varphi_2 = (\varphi_2^{Vertex}: , \varphi_2^{Edge}, \varphi_2^{Label})$, where

$\varphi_1^{Vertex}:$	$\varphi_1^{Edge}:$	$\varphi_1^{Label}:$
$1,2,3 \mapsto 3'$	$(1,2),(1,3) \mapsto (3',3')$	$\gamma_1, \gamma_2 \mapsto \delta_2$

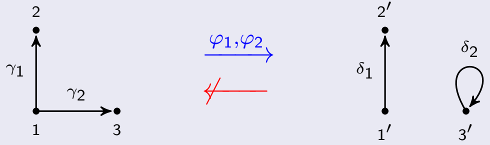


- $\varphi_1 = (\varphi_1^{Vertex}: , \varphi_1^{Edge}, \varphi_1^{Label})$, where

$$\begin{array}{ccc} \varphi_1^{Vertex}: & \varphi_1^{Edge}: & \varphi_1^{Label}: \\ 1 \mapsto 1' & (1,2),(1,3) \mapsto (1',2') & \gamma_1, \gamma_2 \mapsto \delta_1 \\ 2,3 \mapsto 2' & & \end{array}$$

- $\varphi_2 = (\varphi_2^{Vertex}: , \varphi_2^{Edge}, \varphi_2^{Label})$, where

$$\begin{array}{ccc} \varphi_1^{Vertex}: & \varphi_1^{Edge}: & \varphi_1^{Label}: \\ 1,2,3 \mapsto 3' & (1,2),(1,3) \mapsto (3',3') & \gamma_1, \gamma_2 \mapsto \delta_2 \end{array}$$



- $\varphi_1 = (\varphi_1^{Vertex}: , \varphi_1^{Edge}, \varphi_1^{Label})$, where

$\varphi_1^{Vertex}:$	$\varphi_1^{Edge}:$	$\varphi_1^{Label}:$
$1 \mapsto 1'$	$(1,2),(1,3) \mapsto (1',2')$	$\gamma_1, \gamma_2 \mapsto \delta_1$
$2,3 \mapsto 2'$		

- $\varphi_2 = (\varphi_2^{Vertex}: , \varphi_2^{Edge}, \varphi_2^{Label})$, where

$\varphi_1^{Vertex}:$	$\varphi_1^{Edge}:$	$\varphi_1^{Label}:$
$1,2,3 \mapsto 3'$	$(1,2),(1,3) \mapsto (3',3')$	$\gamma_1, \gamma_2 \mapsto \delta_2$

Labelled Multi-Digraphs

The category $\mathcal{LMG}$ of labelled multi-digraphs consists of all directed multi-graphs, $\Gamma = (V, E, \sigma\colon E \to V, \tau\colon E \to V, \lambda\colon E \to \Lambda)$ and morphisms $\varphi\colon \Gamma \to \Gamma'$ where $\varphi = (\varphi^{Vertex}\colon V \to V', \varphi^{Edge}\colon E \to E')$ such that for every $e \in E$,

$$\begin{aligned} \sigma'(\varphi^{Edge}(e)) &= \varphi^{Vertex}(\sigma(e)), \\ \tau'(\varphi^{Edge}(e)) &= \varphi^{Vertex}(\tau(e)), \\ \lambda'(\varphi^{Edge}(e)) &= \varphi^{Label}(\lambda(e)). \end{aligned} \tag{3}$$

- There may be multiple edges in the same direction between nodes.
- There may be more than one edge with the same source and target, unlike for digraphs and labelled digraphs.

Deterministic Transition Graph

$\mathcal{DTG}$ is defined as a full subcategory of labelled multi-digraphs. In fact, a deterministic transition graph is a labelled multi-digraph, $\Gamma = (V, E, \sigma\colon E \to V, \tau\colon E \to V, \lambda\colon E \to \Lambda)$, where

$$(\sigma(e) = \sigma(e') \wedge \lambda(e) = \lambda(e')) \Rightarrow e = e'. \quad (4)$$

- From each vertex there is at most one edge with any given label.
- for all Γ and Γ' in $\mathcal{DTG}$, the morphisms from Γ to Γ' are exactly the $\mathcal{LMG}$-morphisms:

$$\mathcal{DTG}[\Gamma, \Gamma'] = \mathcal{LMG}[\Gamma, \Gamma']. \quad (5)$$

- The initial and terminal objects are the same as in $\mathcal{LMG}$.

Fully-Defined Deterministic Transition Graph

A Fully-Defined Deterministic Transition Graph is a Deterministic Transition Graph $\Gamma = (V, E, \sigma \colon E \to V, \tau \colon E \to V, \lambda \colon E \to \Lambda)$, where

$$\forall v \in V \; \forall x \in \Lambda \;\; \exists e \in E \text{ such that } \sigma(e) = v \text{ and } \lambda(e) = x. \quad (6)$$

- From each vertex there is an outgoing edge with any given label $x \in \Lambda$.
- $\mathcal{FDTG}$ is the full subcategory of $\mathcal{DTG}$ whose objects are the fully-defined deterministic transition graphs and the morphisms are the same.

$\mathcal{LMG}'$

We consider the full subcategory $\mathcal{LMG}'$ of $\mathcal{LMG}$ whose objects $\Gamma = (V, E, \Lambda, \sigma, \tau, \lambda)$ are labelled multi-digraphs those in which distinct parallel edges have distinct labels, i.e. for all edges e, e',

$$\sigma(e) = \sigma(e') \text{ and } \tau(e) = \tau(e') \text{ and } \lambda(e) = \lambda(e') \implies e = e'. \tag{7}$$

- These directed labelled multi-graphs can be regarded as automata.

collapsing

The functor *collapse*: $\mathcal{LMG} \to \mathcal{LMG}'$ is defined by collapsing parallel edges with the same label to a single edge with that same label.

- The functor *collapse* is left adjoint to the inclusion functor $\iota\colon \mathcal{LMG}' \to \mathcal{LMG}$

Theorem

Labelled multi-digraphs in which parallel edges are distinctly labelled, $\mathcal{LMG}'$, comprise a reflective subcategory of labelled multi-graphs $\mathcal{LMG}$. That is, the inclusion functor ι has a left adjoint collapse:

$$\mathcal{LMG}[\Gamma, \iota(\Gamma')] \cong \mathcal{LMG}'[\mathit{collapse}(\Gamma), \Gamma'] \tag{8}$$

- $\lim_{\mathcal{LMG}'} \mathit{collapse}(\Delta) \ncong \mathit{collapse}(\lim_{\mathcal{LMG}} \Delta)$.
- $\mathit{collapse}(\mathrm{colim}_{\mathcal{LMG}}(\Delta)) = \mathrm{colim}_{\mathcal{LMG}'} \mathit{collapse}(\Delta)$.
- In particular, if Δ has no graphs having parallel edges with the same label, then

$$\lim_{\mathcal{LMG}} \Delta = \lim_{\mathcal{LMG}'} \Delta.$$

$$\mathit{collapse}(\mathrm{colim}_{\mathcal{LMG}} \Delta) = \mathrm{colim}_{\mathcal{LMG}'} \Delta.$$

Canonical Form

For each labelled multi-digraph in $\mathcal{LMG}'$ there is a canonical form which is defined by renaming the edges by the **canonical form functor** $C \colon \mathcal{LMG}' \to \mathcal{LMG}'$ defined with $\Gamma = (V, E, \Lambda, \sigma, \tau, \lambda) \mapsto \Gamma' = (V, E', \Lambda, \sigma', \tau', \lambda')$ where $E' \subseteq V \times \Sigma \times V$ and for all $v \in V, e \in E$ and $x \in \Sigma$,

$$\begin{aligned} v &\mapsto v, \\ e &\mapsto e' = (\sigma(e), \lambda(e), \tau(e)), \\ x &\mapsto x, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \sigma'(e') &= \sigma(e), \\ \tau'(e') &= \tau(e), \\ \lambda'(e') &= \lambda(e). \end{aligned} \tag{10}$$

Using this renaming of edges, it is clear how C is defined on morphisms.

- **labelled directed multigraphs in canonical form** $\mathcal{LMG}'' = C(\mathcal{LMG}')$.
- There is a natural isomorphism $\eta\colon Id \dot{\to} C$ since we can rename the edges e of as labelled multi-digraphs in $\mathcal{LMG}'$ with triples $(\sigma(e), \lambda(e), \tau(e))$ up to isomorphism.
- Limits and colimits for diagrams in $\mathcal{LMG}'' = C(\mathcal{LMG}')$ are the same as they are when taken in $\mathcal{LMG}$.

Proposition

$C(\Gamma) \cong \Gamma$ for all Γ in $\mathcal{LMG}'$. And

$$\begin{array}{ccccccc} \mathcal{FDTG}'' & \subset & \mathcal{DTG}'' & \subset & \mathcal{LMG}'' & & \\ \| & & \| & & \| & & \\ C(\mathcal{FDTG}) & \subset & C(\mathcal{DTG}) & \subset & C(\mathcal{LMG}') & & \\ \bigcap & & \bigcap & & \bigcap & & \\ \mathcal{FDTG} & \subset & \mathcal{DTG} & \subset & \mathcal{LMG}' & \subset & \mathcal{LMG} \end{array} \quad (11)$$

Automata Category

$\mathcal{A}$**utom**, $\mathcal{A}$**utom**$^{\mathbf{det}}$, $\mathcal{A}$**utom**$_{\mathbf{(c)}}$,and $\mathcal{A}$**utom**$^{\mathbf{det}}_{\mathbf{(c)}}$, consist of all partial non-deteministic automata, all partial deterministic automata, all complete automata, and complete deterministic automata, respectively. In each case, a morphism $\varphi \colon A \to A'$ is defined as a pair $\varphi = (\varphi^{State} \colon Q \to Q', \varphi^{Input} \colon \Sigma \to \Sigma')$ such that

$$\varphi^{State}(\delta(q,x)) \subseteq \delta'(\varphi^{State}(q), \varphi^{Input}(x)). \tag{12}$$

- Here φ^{State} and φ^{Input} are functions (in particular, these are fully defined and single-valued), and composition of morphisms is component-wise composition of functions.
- we regard the left hand side as the empty set whenever $\delta(q,x)$ is undefined.
- $$\mathcal{A}\mathbf{utom}^{\mathbf{det}}_{\mathbf{(c)}} \subset \mathcal{A}\mathbf{utom}^{\mathbf{det}} \subset \mathcal{A}\mathbf{utom}.$$
  $$\mathcal{A}\mathbf{utom}^{\mathbf{det}}_{\mathbf{(c)}} \subset \mathcal{A}\mathbf{utom}_{\mathbf{(c)}} \subset \mathcal{A}\mathbf{utom}.$$

Associated Automaton to a Labelled Directed Multigraph

For a labelled directed multigraph $\Gamma = (V, E, \sigma, \tau, \lambda)$, define its **associated automaton** as

$$\mathcal{A}(\Gamma) = (\underbrace{V}_{\text{states}}, \underbrace{\Lambda}_{\text{input alphabet}}, \underbrace{\delta \colon V \times \Lambda \to V}_{\text{multivalued partial function}}), \tag{13}$$

- $\delta(v, x) = \{v' \in V \mid \exists e \in E \text{ where } \sigma(e) = v, \tau(e) = v' \text{ and } \lambda(e) = x\}$.
- If $\delta(v, x)$ is empty, we say $\delta(v, x)$ is undefined.
- This defines a functor $\mathcal{A}$ from the $\mathcal{LMG}$ to $\mathcal{A}utom$ as follows:

$$\begin{aligned} \varphi^{\mathcal{A}^{State}} \colon V &\to V' \\ v &\mapsto \varphi^{Vertex}(v), \\ \\ \varphi^{\mathcal{A}^{Input}} \colon \Lambda &\to \Lambda' \\ x &\mapsto \varphi^{Label}(x). \end{aligned} \tag{14}$$

BIOMICS

Associated Labelled Directed Multigraph of an Automaton

Given an automaton $A = (Q, \Sigma, \delta\colon Q \times \Sigma \to Q)$, with δ possibly multiple-valued and partial. Define $\Gamma(A)$ to be the labelled directed multiple graph $\Gamma(A) = (Q, E(A), \Sigma, \sigma, \tau, \lambda)$ with edges

$$E(A) = \{(q, x, q')\colon q' \in \delta(q, x)\},$$

where $\sigma(q, x, q') = q$, $\tau(q, x, q') = q'$, and $\lambda(q, x, q') = x$.

- This defines a functor $\mathcal{A}$ from the $\mathcal{A}utom$ to $\mathcal{LMG}$ as follows:

$$\begin{aligned}
\Gamma(\varphi)^{Vertex}\colon & Q \to Q' \\
& q \mapsto \varphi^{State}(q), \\
\Gamma(\varphi)^{Edge}\colon & E(A) \to E(A') \\
& (q, x, q') \mapsto (\varphi^{State}(q), \varphi^{Input}(x), \varphi^{State}(q')), \\
\Gamma(\varphi)^{Label}\colon & \Sigma \to \Sigma' \\
& x \mapsto \varphi^{Input}(x).
\end{aligned}$$

Theorem (Labelled Multi-Digraphs Correspond to Nondeterministic Partial Automata)

The category of labelled directed multigraphs in canoncial form $\mathcal{LMG}''$ is isomorphic to the category of partial nondeterministic automata $\mathcal{A}utom$.

Theorem (Labelled Multi-digraph–Automata Natural Correspondences)

The associated automaton functor from labelled mult-digraphs to partial, nondeterministic automata $\mathcal{A}\colon \mathcal{LMG} \to \mathcal{A}utom$ restricts to functors $\mathcal{A}\colon \mathcal{LMG}' \to \mathcal{A}utom$, $\mathcal{A}\colon \mathcal{LMG}'' \to \mathcal{A}utom$, $\mathcal{A}\colon \mathcal{DTG}'' \to \mathcal{A}utom^{det}$ and $\mathcal{A}\colon \mathcal{FDTG}'' \to \mathcal{A}utom^{det}_{(c)}$. In the last three cases, Γ is an inverse functor. The $\mathcal{A} \circ \Gamma$ is the identity functor on automata. $\Gamma \circ \mathcal{A}$ is the identity functor on those labelled multi-digraphs in canoncial form that have no distinct parallel edges with the same label.

BIOMICS

- The following diagram commutes

$$\begin{array}{ccccc} \mathcal{LMG}'' & \subset & \mathcal{LMG}' & \subset & \mathcal{LMG} \\ \| & & \| & & \| \\ \mathcal{LMG}'' & \overset{C}{\twoheadleftarrow} & \mathcal{LMG}' & \overset{collapse}{\twoheadleftarrow} & \mathcal{LMG} \end{array} \tag{15}$$

- C renames edges in the canonical form: (source vertex, label, target vertex), and we have

$$\mathcal{A} \circ collapse = \mathcal{A} \circ C \circ collapse = \mathcal{A} \colon \mathcal{LMG} \to \mathcal{A}utom \tag{16}$$

- We have isomorphisms of categories given by the functors Γ and $\mathcal{A}$:

$$\begin{array}{ccccc} \mathcal{FDTG}'' & \subset & \mathcal{DTG}'' & \subset & \mathcal{LMG}'' \\ \wr\| & & \wr\| & & \wr\| \\ \mathcal{A}utom^{det}_{(c)} & \subset & \mathcal{A}utom^{det} & \subset & \mathcal{A}utom, \end{array} \tag{17}$$

where $\mathcal{LMG}''$ consists of labelled directed multigraphs with edges e in canonical form $e = (\sigma(e), \lambda(e), \tau(e))$.

Labelled Digraph Pushout

Let $\Gamma = (V_\Gamma, E_\Gamma, \lambda_\Gamma \colon E_\Gamma \to \Lambda_\Gamma)$, $\Delta = (V_\Delta, E_\Delta, \lambda_\Delta \colon E_\Delta \to \Lambda_\Delta)$, $B = (V_B, E_B, \lambda_B \colon E_B \to \Lambda_B)$ be labelled digraphs, $\alpha \colon B \to \Gamma$ and $\beta \colon B \to \Delta$ be two labelled digraph morphisms. Then, define the labelled digraph $\Phi = (V_\Phi, E_\Phi, \lambda_\Phi \colon E_\Phi \to \Lambda_\Phi)$ by

$$V_\Phi = V_\Gamma \sqcup V_\Delta / \equiv,$$

where $\equiv$ is the equivalence relation generated by the relation R on $V_\Gamma \sqcup V_\delta$,

$$v_\gamma \; R \; v_\delta \Leftrightarrow \exists v_b \text{ such that } \alpha^{Vertex}(v_b) = v_\gamma \text{ and } \beta^{Vertex}(v_b) = v_\delta,$$

$$E_\Phi = \{([v], [w]) \mid \exists (v', w') \in E_\Gamma \sqcup E_\Delta, [v] = [v'] \text{ and } [w] = [w']\},$$

Labelled Digraph Pushout

$$\Lambda_\Phi = \Lambda_\Gamma \sqcup \Lambda_\Delta / \equiv,$$

where $\equiv$ is the equivalence relation generated by the relation R on $\Lambda_\Gamma \sqcup \Lambda_\Delta$,

$$\gamma \; R \; \delta \Leftrightarrow \exists b \text{ such that } \alpha^{Label}(b) = \gamma \text{ and } \beta^{Label}(b) = \delta, or$$
$$\exists([v],[w]) = ([v'],[w']) \in E_\Phi \text{ such that } \lambda_i(v,w) = \gamma \text{ and}$$
$$\lambda_j(v',w') = \delta, i,j \in \{\Gamma, \Delta\},$$

and the label function $\lambda_\Phi \colon E_\Phi \to \Lambda_\Phi$ with,

$$([v],[w]) \mapsto [\lambda_i(v',w')], \text{ where } \exists(v',w') \in E_i \text{ such that}$$

$$([v],[w]) = ([v'],[w']) \text{ and } i \in \{\Gamma, \Delta\}.$$

Labelled Digraph Pushout

Also, define two digraph morphisms $i_K\colon K \to \Phi$ where $K \in \{\Gamma, \Delta\}$ by

$$\begin{aligned} i_K^{Vertex} &: V_K \to V_\Phi \text{ with } v \mapsto [v], \\ i_K^{Edge} &: E_K \to E_\Phi \text{ with } e = (v_1, v_2) \mapsto (i_K^{Vertex}(v_1), i_K^{Vertex}(v_2)), \\ i_K^{Label} &: \Lambda_K \to \Lambda_\Phi \text{ with } x \mapsto [x]. \end{aligned}$$

Then, the labelled digraph Φ with the morphisms i_Γ and i_Δ is the pushout of α and β in the category of labelled digraphs.

$$\begin{array}{ccccc} \Gamma & \xleftarrow{\alpha} & B & \xrightarrow{\beta} & \Delta \\ & \searrow^{i_\Gamma} & & \swarrow_{i_\Delta} & \\ & & \Phi & & \\ & f_\Gamma \searrow & \downarrow \varphi & \swarrow f_\Delta & \\ & & A & & \end{array} \qquad (18)$$

Labelled Multi-Digraph Pushout

Let $\Gamma = (V_\Gamma, E_\Gamma, \sigma_\Gamma, \tau_\Gamma, \lambda_\Gamma)$, $\Delta = (V_\Delta, E_\Delta, \sigma_\Delta, \tau_\Delta, \lambda_\Delta)$, $B = (V_B, E_B, \sigma_B, \tau_B, \lambda_B)$ be labelled multi-digraphs, $\alpha \colon B \to \Gamma$ and $\beta \colon B \to \Delta$ be two labelled multi-digraph morphisms. Then, define the labelled multi-digraph $\Phi = (V_\Phi, E_\Phi, \sigma_\Phi, \tau_\Phi, \lambda_\Phi)$ by

$$K_\Phi = K_\Gamma \sqcup K_\Delta / \equiv$$

where $K \in \{V, E, \Lambda\}$ and $\equiv$ is the equivalence relation generated by the relation R on $K_\Gamma \sqcup K_\delta$,

$$k_\gamma \; R \; k_\delta \Leftrightarrow \exists k_b \text{ such that } \alpha(k_b) = k_\gamma \text{ and } \beta(k_b) = k_\delta,$$

and the label function $\lambda_\Phi \colon E_\Phi \to \Lambda_\Phi$ with,

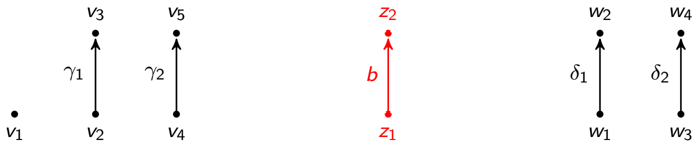
$$e = [e'] \mapsto \begin{cases} [\lambda_\Gamma(e')] & \text{if } e' \in E_\Gamma, \\ [\lambda_\Delta(e')] & \text{if } e' \in E_\Delta, \end{cases}$$

Labelled Multi-Digraph Pushout

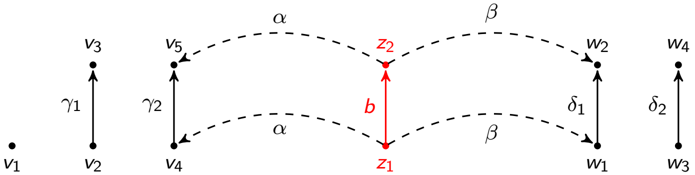
Also, define two digraph morphisms $i_K \colon K \to \Phi$ where $K \in \{\Gamma, \Delta\}$ by

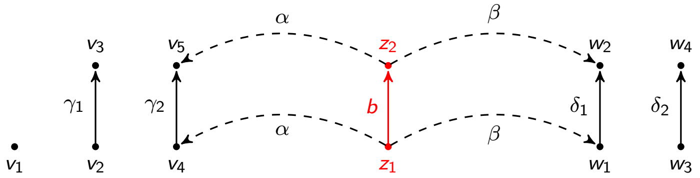
$$
\begin{aligned}
i_K^{Vertex} &: V_K \to V_\Phi \text{ with } v \mapsto [v], \\
i_K^{Edge} &: E_K \to E_\Phi \text{ with } e \mapsto [e], \\
i_K^{Label} &: \Lambda_K \to \Lambda_\Phi \text{ with } x \mapsto [x].
\end{aligned}
$$

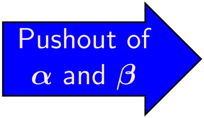
Then, the labelled digraph Φ with the morphisms i_Γ and i_Δ is the pushout of α and β in the category of labelled digraphs.

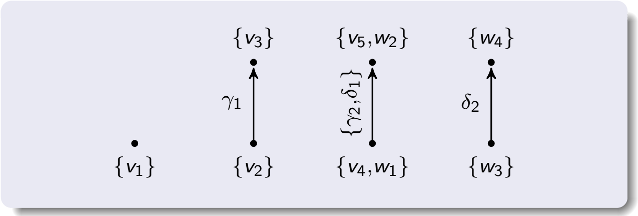


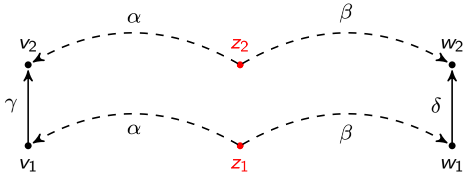
v_3
v_5
γ_1
γ_2
v_1
v_2
v_4
z_2
b
z_1
w_2
w_4
δ_1
δ_2
w_1
w_3


α
β
v_3
v_5
z_2
w_2
w_4
γ_1
γ_2
b
δ_1
δ_2
α
β
v_1
v_2
v_4
z_1
w_1
w_3

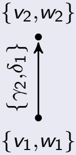

α
β
v_3
v_5
z_2
w_2
w_4
γ_1
γ_2
b
δ_1
δ_2
α
β
v_1
v_2
v_4
z_1
w_1
w_3


Pushout of
α and β

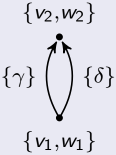

$\{v_3\}$
$\{v_5, w_2\}$
$\{w_4\}$
γ_1
$\{\gamma_2, \delta_1\}$
δ_2
$\{v_1\}$
$\{v_2\}$
$\{v_4, w_1\}$
$\{w_3\}$



Pushout in Labelled Digraph Category



Pushout in Labelled Multi-Digraph Category



Thank You!