~~Hierarchical Decomposition: An Intuitive Approach~~
Holonomy Decomposition: Finally Some Explanations

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Big, abstract ideas and how they appear in the holonomy decomposition of finite transformation semigroups.

abstract idea	**holonomy decomposition**
approximation	?
compression	?
hierarchical structure	?

approximation

Using a sequence of simpler systems, in which the members are increasingly similar to but not the same as the original system to be approximated.

- a curve imitated by increasingly more straight line segments with smaller and smaller length

compression

When several copies of the same component exist in a system, then it is enough to deal with a single copy of the components and the locations of the copies.

- units with the same floorplan in a towerhouse
- subgroup and a transversal (coset representatives)

hierarchical structure

Control information goes in one direction only – component connections are defined by a directed acyclic graph, but in practice we do rooted trees.

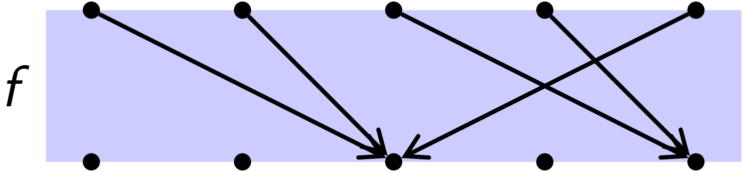
- arithmetic
- robotic arm

statement

In the holonomy decomposition of finite transformation semigroups we *approximate* states and transformations, then we *compress* approximation data and and finally put the encoded form into a hierarchical structure (cascade product).

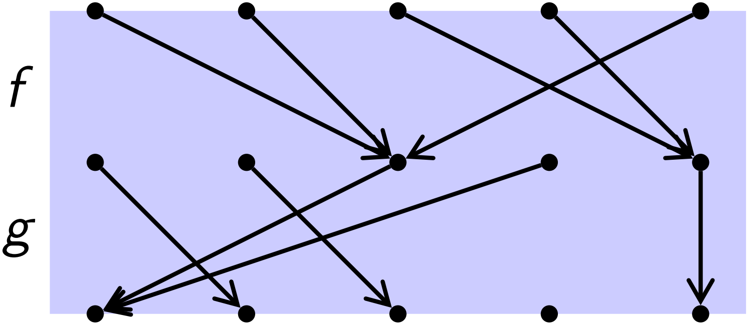
transformation

A *transformation* is a function from a set to itself, $f : X \to X$.



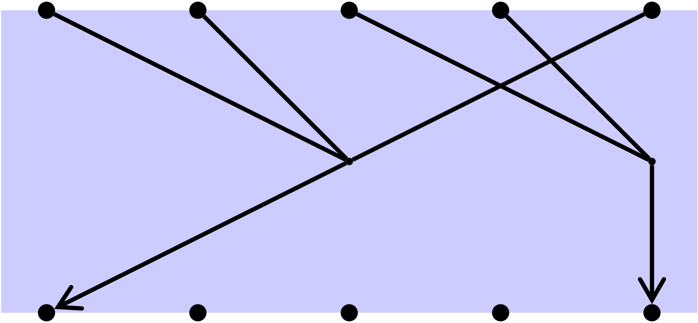
$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 5 & 5 & 3 \end{pmatrix}, \; X = \{1, 2, 3, 4, 5\}$$

We can combine transformations by stacking them...

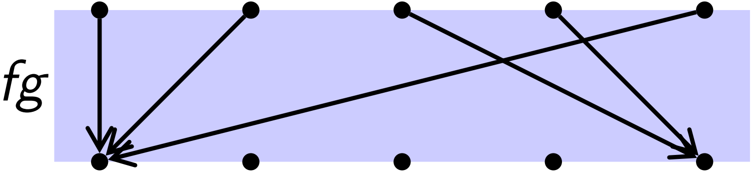


$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 5 & 5 & 3 \end{pmatrix}, g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 1 & 5 \end{pmatrix}$$

...and then by following the connected lines...



...we get another transformation.



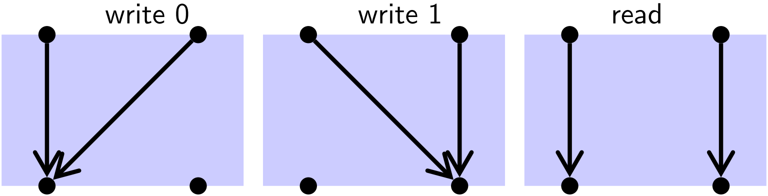
$$\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 3 & 3 & 5 & 5 & 3\end{pmatrix} \cdot \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 2 & 3 & 1 & 1 & 5\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 1 & 1 & 5 & 5 & 1\end{pmatrix} = fg$$

Acting on the state set: $X^f = \{3,5\}$, $X^g = \{1,2,3,5\}$, $X^{fg} = \{1,5\}$

transformation semigroup

A *transformation semigroup* (X, S) of degree n is a collection S of transformations of an n-element set X closed under function composition.

Flip-flop, the 1-bit memory semigroup.



So these are computational devices... $\approx$ automata

btw.

	#subsemigroups	#conjugacy classes	#isomorphism classes
$\mathcal{T}_0$	1	1 (0)	1
$\mathcal{T}_1$	2	2 (1)	2
$\mathcal{T}_2$	10	8 (2)	7
$\mathcal{T}_3$	1 299	283 (4)	267
$\mathcal{T}_4$	3 161 965 550	132 069 776 (22)	TBA

A215650, A215651 `http://oeis.org`

`http://arxiv.org/abs/1403.0274`

hierarchical decompositions of finite transformation semigroups

	Natural Numbers	**Finite Automata**
Building Blocks	Primes	Flip-flop Automaton Permutation Automata
Composition	Multiplication	Cascade Product
Precision	Equality	Division, Emulation
Uniqueness	Unique	Different Decompositions

For computer algebra implementations we use the holonomy decomposition.

Approximating states – subset chains

Definition (Chain)

A *chain* c is a set of elements of $\mathcal{P}(X)$ with the property that for any $A, B \in c$ either $A \subseteq B$ or $B \subseteq A$.

Example (subset chains in general)

$X = \{1, 2, 3, 4, 5\}$

	$\{1,2,3,4,5\}$	
$\{1,2,3,4,5\}$	$\{1,3,4,5\}$	$\{2,3,4,5\}$
$\{2,4,5\}$	$\{1,3,4\}$	$\{3,4\}$
$\{4\}$	$\{3,4\}$	
	$\{4\}$	

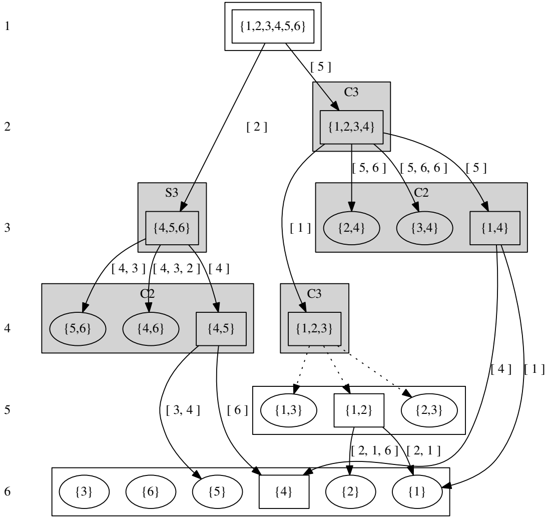
In the holonomy decomposition we act on subset chains, which needs a bit of orchestration.

extended image set

$\mathcal{I}(X) = \{X \cdot s \mid s \in S\}$

$\mathcal{I}'(X) = \mathcal{I}(X) \cup \big\{\{x\} \mid x \in X\big\} \cup X$

maximal chains in $\mathcal{I}'(X)$ are tile chains



tiles → reptiles → integers

abstract idea	**holonomy decomposition**
approximation	subset chains, tile chains, acting on tiles
compression	?
hierarchical structure	?

Compression – equivalence classes

Subduction relation

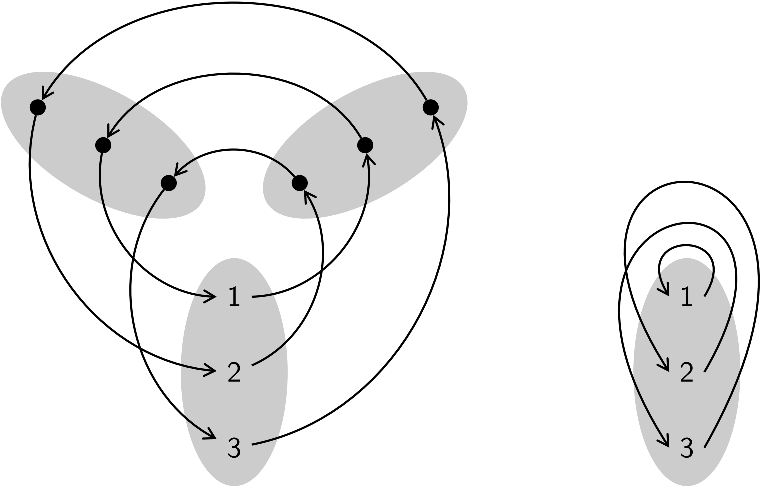
$$P \subseteq_M Q \iff \exists s \in M \text{ such that } P \subseteq Q \cdot s \quad P, Q \in \mathcal{I}(X), \tag{1}$$

i.e. we can transform Q to include P under the action of M. Therefore, subduction is a generalized inclusion, i.e. inclusion under the action of the trivial monoid.

$$P \equiv_M Q \iff P \subseteq_M Q \text{ and } Q \subseteq_M P.$$

$P \equiv_M Q \implies |P| = |Q|$ and $P = Qs$, $Q = Pt$ for some $s, t \in M$.

holonomy groups



(2,3)

holo what?

Definition (holonomy in differential geometry)

Given a smooth closed curve C on a surface M, and picking any point P on that curve, the holonomy of C in M is the angle by which some vector turns as it is parallel transported along the curve C from point P all the way around and back to point P.

(wiktionary.org)

abstract idea	holonomy decomposition
approximation	subset chains, tile chains, acting on tiles
compression	equivalence under action, encoding-decoding
hierarchical structure	?

hierarchical structure – cascade product

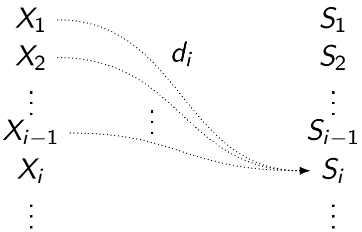
$(X_1, S_1), \ldots, (X_n, S_n)$
A state is an n-tuple $x = (x_1, \ldots, x_n)$, $x_i \in X_i$.

Definition

A *dependency function* d_i of level i for a list of components L is a function

$$d_i : X_1 \times \cdots \times X_{i-1} \to S_i, \quad i \in \mathbf{n}.$$

A dependency function of level i takes $i - 1$ arguments.



Definition

A *transformation cascade* for a given list of components is an n-tuple of dependency functions $(d_1, \ldots, d_n)$, where d_i is a dependency function of level i.

The action of a permutation cascade $d = (d_1, \ldots, d_n)$ on coordinates $x = (x_1, \ldots, x_n)$ for level i is defined by

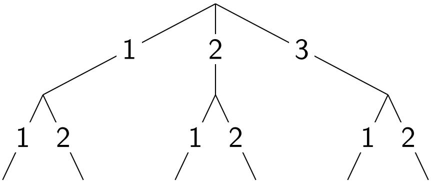
$$x_i^d := x_i^{d_i(x_1, \ldots, x_{i-1})}$$

thus the full action is

$$x^d = (x_1, \ldots, x_n)^{(d_1, \ldots, d_n)} = \left(x_1^{d_1(\varnothing)}, x_2^{d_2(x_1)}, \ldots, x_n^{d_n(x_1, \ldots, x_{n-1})} \right).$$

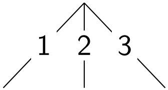
Coordinate tuples as paths in a tree

The set of coordinate tuples $\{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2)\}$ can be represented by a tree.

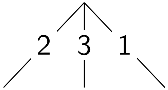


Acting on a tree

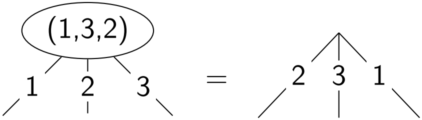
A permutation acting on points can be also thought as acting on a 1-level tree. For instance $(1, 3, 2)$ acting on



yields

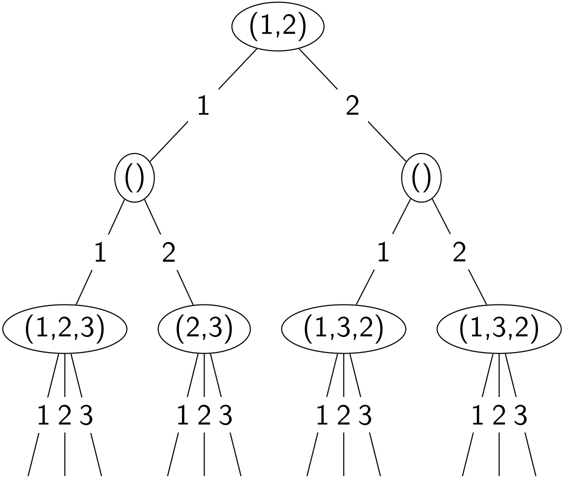


Now we can simply indicate this action on the branches by simply putting the permutation in the root node.

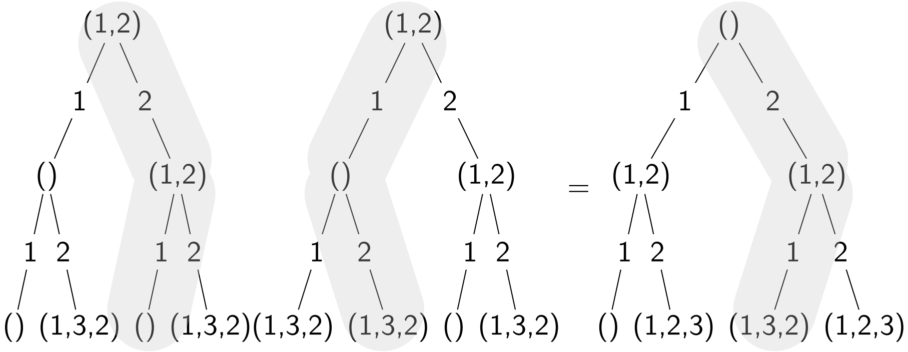


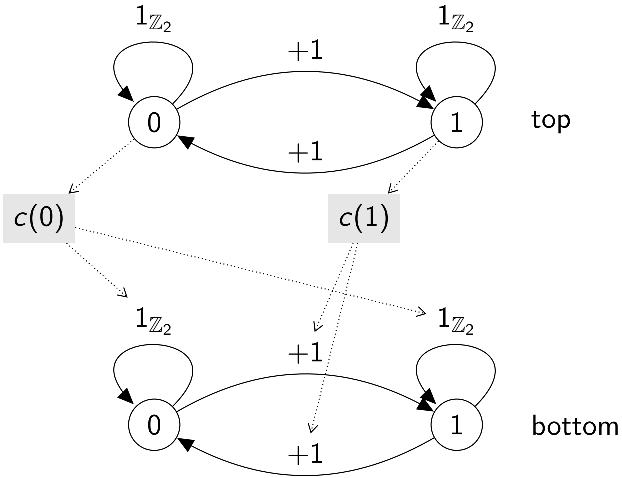
Permutation Cascade

Dependencies: $\varnothing \mapsto (1,2)$, $(1) \mapsto ()$ $(2) \mapsto ()$ $(1,1) \mapsto (1,2,3)$, $(1,2) \mapsto (2,3)$, $(2,1) \mapsto (1,3,2)$, $(2,2) \mapsto (1,3,2)$.



Multiplying Cascades




$1_{\mathbb{Z}_2}$
$1_{\mathbb{Z}_2}$
$+1$
0
1
top
$+1$
$c(0)$
$c(1)$
$1_{\mathbb{Z}_2}$
$1_{\mathbb{Z}_2}$
$+1$
0
1
bottom
$+1$

abstract idea	**holonomy decomposition**
approximation	tiling of subsets, tile chains
compression	equivalence under action, encoding-decoding
hierarchical structure	the placement of components, cascade product

ultimate goals

- what is computable with n states?
- how is the computation done?

Thank You!

Group & semigroup decomposition software:

SgpDec `http://sgpdec.sf.net`

On computational semigroup theory:

`http://compsemi.wordpress.com`