J. Aust. Math. Soc. (First published online 2014), page 1 of 21*
doi:10.1017/S1446788714000111
*Provisional—final page numbers to be inserted when paper edition is published

FINSLER METRIZABLE ISOTROPIC SPRAYS AND HILBERT'S FOURTH PROBLEM

IOAN BUCATARU✉ and ZOLTÁN MUZSNAY

(Received 7 May 2013; accepted 21 January 2014)

Communicated by M. Murray

Abstract

It is well known that a system of homogeneous second-order ordinary differential equations (spray) is necessarily isotropic in order to be metrizable by a Finsler function of scalar flag curvature. In our main result we show that the isotropy condition, together with three other conditions on the Jacobi endomorphism, characterize sprays that are metrizable by Finsler functions of scalar flag curvature. We call these conditions the scalar flag curvature (SFC) test. The proof of the main result provides an algorithm to construct the Finsler function of scalar flag curvature, in the case when a given spray is metrizable. Hilbert's fourth problem asks to determine the Finsler functions with rectilinear geodesics. A Finsler function that is a solution to Hilbert's fourth problem is necessarily of constant or scalar flag curvature. Therefore, we can use the constant flag curvature (CFC) test, which we developed in our previous paper, Bucataru and Muzsnay ['Sprays metrizable by Finsler functions of constant flag curvature', *Differential Geom. Appl.* **31** (3)(2013), 405–415] as well as the SFC test to decide whether or not the projective deformations of a flat spray, which are isotropic, are metrizable by Finsler functions of constant or scalar flag curvature. We show how to use the algorithms provided by the CFC and SFC tests to construct solutions to Hilbert's fourth problem.

2010 *Mathematics subject classification*: primary 53C60; secondary 58B20, 49N45, 58E30.

Keywords and phrases: isotropic sprays, Finsler metrizability, flag curvature, Hilbert's fourth problem.

1. Introduction

Second-order ordinary differential equations (SODEs) are important mathematical objects because they have a large variety of applications in different domains of mathematics, science and engineering [4]. A particularly interesting class of SODE is the one which can be derived from a variational principle. The inverse problem of the calculus of variations (IP) consists of characterizing variational SODEs, which means to determine whether or not a given SODE can be described as the critical point

The work of I.B. has been supported by the Bilateral Cooperation Program Romania-Hungary 672/2013-2014. The work of Z.M. has been supported by EU FET FP7 BIOMICS project, contract number CNECT-318202, by TÉT-12-RO-1-2013-0022 and by TÁMOP-4.2.2.C-11/1/KONV-2012-0010 project.

of a functional. The most significant contribution to this problem is the famous paper of Douglas [19] in which, using Riquier's theory, he classifies variational differential equations with two degrees of freedom. Generalizing his results to higher-dimensional cases is a hard problem because the Euler–Lagrange system is an extremely over-determined partial differential equation (PDE), so in general it has no solution. The integrability conditions of the Euler–Lagrange PDE can be very complex and can change case by case [3, 11, 18, 21, 25, 26, 28]. Therefore, it seems to be impossible to obtain a complete classification of variational SODE in the n-dimensional case, unless we restrict the problem to particular classes of sprays with special curvature properties [6–8, 13, 15, 30].

A special and very interesting problem, within the IP, is known as the Finsler metrizability problem. Here the Lagrangian to search for is the energy function of a Finslerian or a Riemannian metric [24, 27, 32]. Of course, in this problem the given system of SODE must be homogeneous or quadratic. If the corresponding metric exists, then the integral curves of the given SODE are the geodesic curves of the corresponding Finslerian or Riemannian metric. Since the obstructions to the existence of a metric for a given SODE are essentially related to curvature properties of the associated canonical nonlinear connection, it seems to be reasonable to consider SODEs with special curvature properties. Obvious candidates to investigate are Finsler structures with constant or scalar flag curvature. It is therefore natural to formulate the following problem.

Provide the necessary and sufficient conditions that can be used to decide whether or not a given homogeneous system of SODEs represents the Euler–Lagrange equations of a Finsler function of constant flag curvature or scalar flag curvature, respectively. In [13] we solved the first part of the problem by giving a characterization of sprays that are metrizable by Finsler functions of constant flag curvature. We will refer to the conditions of [13, Theorem 4.1] as to the *constant flag curvature test*, or *CFC test* for short. In the present paper we consider the second part of the problem and solve it completely by giving a coordinate free characterization of sprays metrizable by Finsler functions of scalar flag curvature. Our main result can be found in Section 3, where we provide the necessary and sufficient conditions, as tensorial equations on the Jacobi endomorphism, which can be used to decide whether or not a given homogeneous SODE represents the geodesic equations of a Finsler function of scalar curvature. The first necessary condition for a spray to be metrizable by a Finsler function of scalar flag curvature refers to the isotropy, see [29, Lemma 8.2.2]. In Theorem 3.1 we provide three other conditions, which together with the isotropy condition, will characterize the class of sprays that are metrizable by Finsler functions of scalar flag curvature. We will refer to these conditions as to the *scalar flag curvature test*, or *SFC test* for short. The proof of Theorem 3.1 offers, in the case when the SFC test is affirmative, an algorithm to construct the Finsler function of scalar flag curvature that metricizes the given spray. In all of the examples that we provide, we show how to use the proposed algorithm to construct such Finsler functions. The importance of characterizing sprays metrizable by Finsler functions of scalar flag curvature for constructing all systems of ordinary differential equations (ODEs) with vanishing Wilczynski invariants has been discussed recently in [14].

In Section 4 we show that our results for characterizing metrizable sprays lead to a new approach for Hilbert's fourth problem. This problem asks to construct and study the geometries in which the straight line segment is the shortest connection between two points, [1]. Alternatively, one can reformulate the problem as follows: 'given a domain $\Omega \subset \mathbb{R}^n$, determine all (Finsler) metrics on Ω whose geodesics are straight lines', [29, page 191]. Yet another reformulation of the problem requires us to determine projectively flat Finsler metrics [16]. Projectively flat Finsler functions have isotropic geodesic sprays and therefore have constant or scalar flag curvature. Such Finsler metrics, of constant flag curvature where studied in [8, 30]. We use the CFC test, as well as the SFC test, to study when the projective deformations of a flat spray are metrizable. Using these conditions, we show how to construct examples that are solutions to Hilbert's fourth problem, given by Finsler functions of constant and scalar flag curvature, respectively.

In Section 5 we give working examples to show how to use Theorem 3.1 to test whether or not some other sprays are Finsler metrizable, and in the affirmative case how to construct the corresponding Finsler function. By relaxing a regularity condition of Theorem 3.1, we show that we can also characterize sprays that are metrizable by conic pseudo or degenerate Finsler functions.

2. The geometric framework for Finsler metrizability

In this section we present the geometric setting for addressing the Finsler metrizability problem [12, 24, 27, 29, 31]. This geometric setting, which includes connections and curvature, can be derived directly from a given homogeneous SODE using the Frölicher–Nijenhuis formalism (see [9], [21, Ch. 2] and [23, Section 30]).

2.1. Spray, connections and curvature. We consider M a smooth, real and n-dimensional manifold. In this work, all geometric structures are assumed to be smooth. We denote by $C^\infty(M)$ the set of smooth functions on M, by $\mathfrak{X}(M)$ the set of vector fields on M and by $\Lambda^k(M)$ the set of k-forms on M.

For the manifold M, we consider the tangent bundle (TM, π, M) and $(T_0M = TM \setminus \{0\}, \pi, M)$ the tangent bundle with the zero section removed. If (x^i) are local coordinates on the base manifold M, the induced coordinates on the total space TM will be denoted by (x^i, y^i).

The tangent bundle carries some canonical structures, very useful to formulate our geometric framework. One structure is the *vertical subbundle* $VTM = \{\xi \in TTM, (D\pi)\xi = 0\}$, which induces an integrable, n-dimensional distribution $V : u \in TM \to V_u = VTM \cap T_uTM$. Locally, this distribution, which we will refer to as the *vertical distribution*, is spanned by $\{\partial/\partial y^i\}$. Two other structures, defined on TM, are the tangent structure, J, and the Liouville vector field, $\mathbb{C}$, locally given by

$$J = \frac{\partial}{\partial y^i} \otimes dx^i, \quad \mathbb{C} = y^i \frac{\partial}{\partial y^i}.$$

If L is a vector valued l-form on TM, we will denote by i_L and d_L the derivations of degree $(l-1)$ and l, respectively, connected by

$$d_L = [i_L, d] = i_L \circ d - (-1)^{l-1} d \circ i_L.$$

For two vector values forms K and L on TM, of degrees k and l, we consider the Frölicher–Nijenhuis bracket $[K, L]$, which is the vector valued $(k+l)$-form uniquely determined by

$$d_{[K,L]} = [d_K, d_L] = d_k \circ d_L - (-1)^{kl} d_L \circ d_K.$$

For a brief introduction to Frölicher–Nijenhuis theory of derivations we refer to [21, Ch. 2], and for various commutation formulae within this we will use Appendix A of the same book.

The main object of this work is a system of n homogeneous, SODEs, whose coefficients do not depend explicitly on time,

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0. \tag{2.1}$$

For the functions $G^i(x, y)$ we assume that they are positively 2-homogeneous, which means that $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$, for all $\lambda > 0$. By Euler's theorem, the homogeneity condition of the functions G^i is equivalent to $\mathbb{C}(G^i) = 2G^i$.

The system (2.1) can be identified with a special vector field $S \in \mathfrak{X}(T_0M)$ that satisfies $JS = \mathbb{C}$ and the homogeneity condition $[\mathbb{C}, S] = S$. Such a vector field is called a *spray* and it is locally given by

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}. \tag{2.2}$$

If we reparameterize the second-order system (2.1), by preserving the orientation of the parameter, we obtain a new system and hence a new spray $\tilde{S} = S - 2P\mathbb{C}$ (see [4, 29]). The function $P \in C^\infty(T_0M)$ is 1-homogeneous, which means that it satisfies $\mathbb{C}(P) = P$. The two sprays S and $\tilde{S}$ are called *projectively related*, while the function P is called a *projective deformation* of the spray S.

An important geometric structure that can be associated to a spray is that of *nonlinear connection* (horizontal distribution, Ehresmann connection). A nonlinear connection is defined by an n-dimensional distribution $H : u \in TM \to H_u \subset T_uTM$ that is supplementary to the vertical distribution: $T_uTM = H_u \oplus V_u$. It is well known that a spray S induces a nonlinear connection with the corresponding horizontal and vertical projectors given by [20]

$$h = \tfrac{1}{2}(\mathrm{Id} - [S, J]), \quad v = \tfrac{1}{2}(\mathrm{Id} + [S, J]).$$

Locally, the above two projectors can be expressed as follows

$$h = \frac{\delta}{\delta x^i} \otimes dx^i, \quad v = \frac{\partial}{\partial y^i} \otimes \delta y^i$$

where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x, y)\frac{\partial}{\partial y^j}, \ \delta y^i = dy^i + N_j^i(x, y)\, dx^j, \ N_j^i(x, y) = \frac{\partial G^i}{\partial y^j}(x, y).$$

Alternatively, the nonlinear connection induced by a spray S can be characterized in terms of an *almost complex structure*,

$$\mathbb{F} = h \circ [S, h] - J = \frac{\delta}{\delta x^i} \otimes \delta y^i - \frac{\partial}{\partial y^i} \otimes dx^i.$$

It is straightforward to check that $\mathbb{F} \circ J = h$ and $J \circ \mathbb{F} = v$.

The horizontal distribution H is, in general, nonintegrable. The obstruction to its integrability is given by the *curvature tensor* (or the Nijenhuis tensor)

$$R = \frac{1}{2}[h, h] = \frac{1}{2} R^i_{jk} \frac{\partial}{\partial y^i} \otimes dx^j \wedge dx^k, \quad R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}. \tag{2.3}$$

A spray S is said to be *R-flat* if the curvature tensor R, in formula (2.3), vanishes. The terminology was proposed by Shen in [29].

Owing to the homogeneity condition of a spray S, curvature information can be obtained also from the *Jacobi endomorphism*

$$\Phi = v \circ [S, h] = R^i_j \frac{\partial}{\partial y^i} \otimes dx^j, \quad R^i_j = 2\frac{\partial G^i}{\partial x^j} - S(N^i_j) - N^i_k N^k_j. \tag{2.4}$$

The two curvature tensors are related by

$$3R = [J, \Phi], \quad \Phi = i_S R. \tag{2.5}$$

Owing to the above properties, it follows that a spray S is R-flat if and only if the Jacobi endomorphism, Φ, vanishes.

As we will see in this work, important geometric information about the given spray S are encoded in the *Ricci scalar*, $\rho \in C^\infty(T_0M)$ (see [6] and [29, Definition 8.1.7]), which is given by

$$(n-1)\rho = R^i_i = \mathrm{Tr}(\Phi). \tag{2.6}$$

Definition 2.1. A spray S is said to be *isotropic* if there exists a semi-basic 1-form $\alpha \in \Lambda^1(T_0M)$ such that the Jacobi endomorphism can be written as follows

$$\Phi = \rho J - \alpha \otimes \mathbb{C}. \tag{2.7}$$

Owing to the homogeneity condition, for isotropic sprays, the Ricci scalar is given by $\rho = i_S\alpha$. Using formulae (2.5) and (2.7), it can be shown that the class of isotropic sprays can be characterized also in terms of the curvature R of the nonlinear connection [11, Proposition 3.4],

$$3R = (d_J\rho + \alpha) \wedge J - d_J\alpha \otimes \mathbb{C}. \tag{2.8}$$

To complete the geometric setting for studying the Finsler metrizability problem of a spray, we will use also the *Berwald connection*. It is a linear connection on T_0M, for $X, Y \in \mathfrak{X}(T_0M)$, it is given by

$$D_XY = h[vX, hY] + v[hX, vY] + (\mathbb{F} + J)[hX, JY] + J[vX, (\mathbb{F} + J)Y].$$

Locally, the Berwald connection is given by

$$D_{\delta/\delta x^i}\frac{\delta}{\delta x^j} = \frac{\partial N_i^k}{\partial y^j}\frac{\delta}{\delta x^k}, \quad D_{\delta/\delta x^i}\frac{\partial}{\partial y^j} = \frac{\partial N_i^k}{\partial y^j}\frac{\partial}{\partial y^k}.$$
$$D_{\partial/\partial y^i}\frac{\delta}{\delta x^j} = 0, \quad D_{\partial/\partial y^i}\frac{\partial}{\partial y^j} = 0.$$

The Berwald connection provides two covariant derivations, the h- and v-covariant derivations. In this work we will use the h-covariant derivation, whose action on vector fields is given by

$$D^h_X Y = D_{hX}Y = v[hX, vY] + (\mathbb{F} + J)[hX, JY], \quad \forall X, Y \in \mathfrak{X}(T_0M)$$

The Berwald connection has two curvature components. One is the Riemann curvature tensor and it is directly related to the curvature tensor R and the Jacobi endomorphism Φ. Another one is the Berwald curvature [29, Sections 7.1 and 8.1].

A spray S is said to be *flat* if the two curvature components of the Berwald connection vanishes.

The action of the Berwald connection in the direction of the given spray S provides a tensor derivation on T_0M, which is called the *dynamical covariant derivative* [10, Section 3.2]. This is a map $\nabla : \mathfrak{X}(T_0M) \to \mathfrak{X}(T_0M)$, given by

$$\nabla X = h[S, hX] + v[S, vX], \quad \forall X \in \mathfrak{X}(T_0M).$$

We set $\nabla f = Sf$, for all $f \in C^\infty(T_0M)$. By using the Leibniz rule and the requirement that ∇ commutes with the tensor contraction, we can extend the action of ∇ to arbitrary tensor fields and forms on T_0M (see [9]).

2.2. Finsler spaces. In this section, we briefly recall the notion of Finsler functions, as well as some generalizations: conic pseudo-Finsler functions and degenerate Finsler functions. The variational problem for the energy of a Finsler function determines a spray, which is called the geodesic spray. The Finsler metrizability problem requires to decide if a given spray represents the geodesic spray of a Finsler function.

Definition 2.2. A continuous function $F : TM \to \mathbb{R}$ is called a *Finsler function* if it satisfies the following conditions:

(i) F is smooth and strictly positive on T_0M, $F(x, 0) = 0$;
(ii) F is positively homogeneous of order 1 in the fibre coordinates, which means that $F(x, \lambda y) = \lambda F(x, y)$, for all $\lambda > 0$ and $(x, y) \in T_0M$;
(iii) the 2-form dd_JF^2 is a symplectic form on T_0M.

In this work we will allow for some relaxations of the above conditions, regarding the domain of the function as well as the regularity condition (iii). See [4, Sections 1.1.2 and 1.2.1], [8] and [22] for more discussions about the regularity conditions and their relaxation for a Finsler function.

If the function F is defined on some positive conical region $A \subset TM$ and the three conditions of Definition 2.2 are satisfied on $A \cap T_0M$, then we call F a *conic pseudo-Finsler metric*. Moreover, if we replace the regularity condition (iii) by a weaker condition, $\operatorname{rank}(dd_J F^2) \in \{1, \dots, 2n-1\}$ on $A \cap T_0M$, we call F a *degenerate Finsler metric* [22].

Definition 2.3. A spray S is called Finsler metrizable if there exists a Finsler function F such that

$$i_S dd_J F^2 = -dF^2. \tag{2.9}$$

We will also use the metrizability property in a broader sense by calling a spray S conic pseudo or degenerate Finsler metrizable if there exist a conic pseudo or degenerate Finsler function F such that (2.9) is satisfied. If a spray S satisfies (2.9), we call it the *geodesic spray* of the (conic pseudo or degenerate) Finsler function F. It is well known that S is the geodesic spray of such function if and only if satisfies following equation:

$$d_h F^2 = 0. \tag{2.10}$$

Consider S the geodesic spray of some (conic pseudo or degenerate) Finsler function F and let Φ be the Jacobi endomorphism.

Definition 2.4. The Finsler function F is said to be of *scalar flag curvature* if there exists a function $\kappa \in C^\infty(T_0M)$ such that

$$\Phi = \kappa(F^2 J - F d_J F \otimes \mathbb{C}). \tag{2.11}$$

Using the formulae (2.7) and (2.11) it follows that for a Finsler function F, of scalar flag curvature κ, its geodesic spray S is isotropic, with the Ricci scalar $\rho = \kappa F^2$ and the semi-basic 1-form $\alpha = \kappa F d_J F$.

Conversely, it can be shown that if an isotropic spray S is metrizable by a Finsler function F, then F is necessarily of scalar flag curvature. See [29, Lemma 8.2.2] or the first implication in [13, Theorem 4.2] for an alternative proof. One can conclude the above considerations as follows.

Remark. For a Finsler function, its geodesic spray is isotropic if and only if the Finsler function is of scalar flag curvature.

3. Sprays metrizable by Finsler functions of scalar curvature

The problem we want to address in this paper is the following: provide the necessary and sufficient conditions for a sprays S to be metrizable by a Finsler function of scalar flag curvature. The discussion from the end of the previous section restricts the class of sprays to start with to the class of isotropic sprays. Alternative formulations of the conditions we use in the next theorem were proposed first in [21, Theorem 7.2], in the analytic case, to decide when a nonflat isotropic spray is variational, by discussing the formal integrability of an associated partial differential operator. However, the next theorem, will provide an algorithm to construct the Finsler function that metricizes a given spray, in the case when such spray is variational. Moreover, the differentiability assumption we use in the next theorem is weaker, all geometric structure we use are smooth, not necessarily analytic. Next theorem extends the CFC test of Theorem 4.1 in [13], where we characterize sprays metrizable by Finsler functions of constant flag curvature.

Theorem 3.1 (SFC test). *Consider S a spray of nonvanishing Ricci scalar. The spray S is metrizable by a Finsler function F, of nonvanishing scalar flag curvature, if and only if:*

(i) *S is isotropic;*
(ii) *$d_J(\alpha/\rho) = 0$;*
(iii) *$D^h(\alpha/\rho) = 0$;*
(iv) *$d(\alpha/\rho) + 2i_{\mathbb{F}}\alpha/\rho \wedge \alpha/\rho$ is a symplectic form on T_0M.*

Proof. We assume that the spray S is metrizable by a Finsler function F of scalar flag curvature κ and we will prove that the four conditions (i)–(iv) are necessary.

Since the Jacobi endomorphism Φ is given by formula (2.11), as we discussed already, it follows that S is isotropic, and hence condition (i) is satisfied.

The semi-basic 1-form α and the Ricci scalar ρ are given by

$$\alpha = \kappa F d_J F, \quad \rho = \kappa F^2. \tag{3.1}$$

It follows that $\alpha/\rho = d_J F/F$ and therefore $d_J(\alpha/\rho) = 0$, which means that the condition (ii) is satisfied.

Since S is the geodesic spray of the Finsler function F, it follows from first formula (2.10) that $d_h F = 0$. Therefore, $D_{hX}F = (hX)(F) = (d_hF)(X) = 0$ and $D_{hX}d_JF = 0$. It follows that $D^h(\alpha/\rho) = 0$ and hence the condition (iii) is also satisfied.

We check now the regularity condition (iv). Using $d_hF = 0$ and $J \circ \mathbb{F} = v$, we obtain

$$i_{\mathbb{F}}\frac{\alpha}{\rho} = i_{\mathbb{F}}\frac{1}{F}d_JF = \frac{1}{F}d_vF = \frac{1}{F}\,dF.$$

Therefore, using the regularity of the Finsler function F, it follows that

$$d\left(\frac{\alpha}{\rho}\right) + 2i_{\mathbb{F}}\frac{\alpha}{\rho} \wedge \frac{\alpha}{\rho} = d\left(\frac{d_JF}{F}\right) + \frac{2}{F^2}\,dF \wedge d_JF = \frac{1}{2F^2}\,dd_JF^2$$

is a symplectic form on T_0M.

Let us prove now the sufficiency of the four conditions (i)–(iv).

Consider S a spray that satisfies all four conditions (i)–(iv). First condition (i) says that the spray S is isotropic, which means that its Jacobi endomorphism Φ is given by formula (2.7). Next three conditions (ii)–(iv) refer to the semi-basic 1-form α and the Ricci scalar ρ, which enter into the expression (2.7) of the Jacobi endomorphism Φ.

From condition (ii) we have that the semi-basic 1-form α/ρ is a d_J-closed 1-form. Since the tangent structure J is integrable, it follows that $[J, J] = 0$ and hence $2d_J^2 = d_{[J,J]} = 0$. Therefore, using a Poincaré-type Lemma for the differential operator d_J, it follows that, locally, α/ρ is a d_J-exact 1-form. It follows that there exists a function f, locally defined on T_0M, such that

$$\frac{1}{\rho}\alpha = d_J f = \frac{\partial f}{\partial y^i}\, dx^i. \tag{3.2}$$

Note that this function f is not unique, it is given up to an arbitrary basic function $a \in C^\infty(M)$. We will prove that using this function f and a corresponding basic function a, we can construct a Finsler function $F = \exp(f - a)$, of scalar flag curvature, which metricizes the given spray S.

Using the commutation rule for i_S and d_J, see [21, Appendix A], we have

$$\mathbb{C}(f) = i_S d_J f = i_S \frac{\alpha}{\rho} = 1. \tag{3.3}$$

Using the condition (ii) of the theorem, and the form (2.8) of the curvature tensor R, we obtain

$$\begin{aligned} 3d_R f &= (d_J\rho + \alpha) \wedge d_J f - \mathbb{C}(f) d_J\alpha \\ &= (d_J\rho + \alpha) \wedge \frac{\alpha}{\rho} - d_J\alpha = -\rho d_J\left(\frac{\alpha}{\rho}\right) = 0. \end{aligned} \tag{3.4}$$

The condition (iii) of the theorem can be written locally as follows

$$D_{\delta/\delta x^i} \frac{\partial f}{\partial y^j} = \frac{\partial}{\partial y^j}\left(\frac{\delta f}{\delta x^i}\right) = 0, \tag{3.5}$$

which means that the components $\omega_i = \delta f/\delta x^i$ are independent of the fibre coordinates. In other words

$$\omega = d_h f = \frac{\delta f}{\delta x^i}\, dx^i, \tag{3.6}$$

is a basic 1-form on T_0M. Using formula (3.4) we have

$$0 = d_R f = d_h^2 f = d_h(d_h f) = \frac{1}{2}\left(\frac{\partial\omega_i}{\partial x^j} - \frac{\partial\omega_j}{\partial x^i}\right) dx^i \wedge dx^j = d(d_h f). \tag{3.7}$$

It follows that the basic 1-form $d_h f \in \Lambda^1(M)$ is closed and hence it is locally exact. Therefore, there exists a function a, which is locally defined on M, such that

$$d_h f = da = d_h a. \tag{3.8}$$

We will prove now that the function

$$F = \exp(f - a), \tag{3.9}$$

locally defined on T_0M, is a Finsler function of scalar flag curvature, whose geodesic spray is the given spray S. Depending on the domain of the two functions f and a, the function F might be a conic pseudo-Finsler function.

From formula (3.3), we have that $\mathbb{C}(F) = \exp(f - a)\mathbb{C}(f) = F$, which means that F is 1-homogeneous. Using formula (3.8), we obtain that

$$d_h F = \exp(f - a) d_h(f - a) = 0. \tag{3.10}$$

The semi-basic 1-form α/ρ, which is given by formula (3.2), can be expressed in terms of the function F, given by formula (3.9), as follows

$$\frac{\alpha}{\rho} = \frac{d_J F}{F}.$$

We use now formula (3.10) and obtain

$$d\left(\frac{\alpha}{\rho}\right) + 2i_{\mathbb{F}}\frac{\alpha}{\rho} \wedge \frac{\alpha}{\rho} = \frac{1}{F^2} dd_J F^2. \tag{3.11}$$

The last condition of the theorem assures that $dd_J F^2$ is a symplectic form and hence F is a Finsler function. Owing to formula (3.10), we obtain that S is the geodesic spray of the Finsler function F.

To complete the proof, we have to show now that F has nonvanishing scalar flag curvature. Since the Finsler function F is given by formula (3.9), we have that $F > 0$ on T_0M and we may consider the function

$$\kappa = \frac{\rho}{F^2}. \tag{3.12}$$

It follows that the semi-basic 1-form α is given by

$$\alpha = \frac{\rho}{F} d_J F = \kappa F d_J F. \tag{3.13}$$

Since the Ricci scalar does not vanish, it follows that the function κ has the same property. The last two formulae (3.12) and (3.13) show that the Jacobi endomorphism Φ, of the geodesic spray S of the Finsler function F, is given by formula (2.11). Therefore, the Finsler function F has nonvanishing scalar flag curvature κ. □

We can replace the regularity condition (iv) of Theorem 3.1 by a weaker condition and require that $\operatorname{rank}(d(\alpha/\rho) + 2i_{\mathbb{F}}\alpha/\rho \wedge \alpha/\rho) \neq 0$ on some conical region in T_0M. In this case the theorem provides a characterization for sprays metrizable by conic pseudo or degenerate Finsler function. We consider two examples of such sprays in Section 5.

For dimensions greater than two, Theorem 3.1 does not address the Finsler metrizability problem in its most general context. The cases that are not covered by this theorem refer to nonisotropic sprays that are metrizable by Finsler functions.

However, in the two-dimensional case, the SFC test of Theorem 3.1 covers the Finsler metrizability problem in the most general case. This is due to the fact that any two-dimensional spray is isotropic and, therefore, the Finsler metrizability problem is equivalent to the metrizability by a Finsler function of scalar flag curvature. For the two-dimensional case, in [7], Berwald provides the necessary and sufficient conditions, in terms of the curvature scalars, such that the extremals of a Finsler space are rectilinear.

Dimension two is also important due to Douglas' work [19], where the inverse problem of the calculus of variation for two degrees of freedom is solved completely. For dimension two, our Theorem 3.1 corresponds to case II, in Douglas' classification. In order to see this aspect we make use of the modern reformulation of Douglas' classification from [18]. For $X = S = hS$, the condition (iii) of Theorem 3.1 implies $\nabla(\alpha/\rho) = 0$. For an isotropic spray S, we write its Jacobi endomorphism as follows

$$\Phi = \rho\Big(J - \frac{\alpha}{\rho} \otimes \mathbb{C}\Big).$$

Using some properties of the dynamical covariant derivative, [10], $\nabla J = 0$ and $\nabla \mathbb{C} = 0$, it follows

$$\nabla\Phi = S(\rho)\Big(J - \frac{\alpha}{\rho} \otimes \mathbb{C}\Big) = \frac{S(\rho)}{\rho}\Phi.$$

The above formula shows that $\nabla\Phi$ and Φ are linear dependent, which is case II in Douglas' analysis.

The importance of characterizing sprays that are metrizable by Finsler functions of scalar flag curvature was discussed recently in [14] since it will allow to 'construct all systems of ODEs with vanishing Wilczynski invariants'.

4. Hilbert's fourth problem

'Hilbert's fourth problem asks to construct and study the geometries in which the straight line segment is the shortest connection between two points' [1]. Alternatively, the problem can be reformulated as follows: 'given a domain $\Omega \subset \mathbb{R}^n$, determine all (Finsler) metrics on Ω whose geodesics are straight lines' [29, page 191]. These Finsler metrics are projectively flat and can be studied using different techniques [16, 17, 30]. All such Finsler functions have constant or scalar flag curvature. Therefore, we can use the conditions of [13, Theorem 4.1] and Theorem 3.1 to test when a projectively flat spray is Finsler metrizable. For such sprays we use the algorithms provided in the proofs of [13, Theorem 4.1] and Theorem 3.1 to construct solutions to Hilbert's fourth problem.

We start with S_0, the flat spray on some domain $\Omega \subset \mathbb{R}^n$. A projective deformation $S = S_0 - 2P\mathbb{C}$ leads to a new spray that is isotropic. In the case when the spray S satisfies either the CFC test or the SFC test, then S is the geodesic spray of a Finsler function of constant or scalar flag curvature. In this way we provide a method to construct Finsler functions of constant or scalar flag curvature with rectilinear geodesics.

Consider a domain $\Omega \subset \mathbb{R}^n$ and let $S_0 \in \mathfrak{X}(\Omega \times \mathbb{R}^n)$ be the flat spray. We will study now, when a projective deformation

$$S = S_0 - 2P\mathbb{C} = y^i \frac{\partial}{\partial x^i} - 2Py^i \frac{\partial}{\partial y^i}, \tag{4.1}$$

for a 1-homogeneous function $P \in C^\infty(\Omega \times \mathbb{R}^n \setminus \{0\})$, leads to a metrizable spray S by a Finsler function F of constant or scalar flag curvature. Such a Finsler function F will be a solution to Hilbert's fourth problem.

Using the formulae [12, (4.8)], the Jacobi endomorphism of the new spray S is given by

$$\Phi = (P^2 - S_0 P)J - (P d_J P + d_J(S_0 P) - 3 d_{h_0} P) \otimes \mathbb{C}. \tag{4.2}$$

It follows that the spray S is isotropic, the Ricci scalar, ρ, and the semi-basic form α are given by

$$\rho = P^2 - S_0 P, \quad \alpha = P d_J P + d_J(S_0 P) - 3 d_{h_0} P. \tag{4.3}$$

From the above formula it follows that

$$d_J \alpha = -3 d_J d_{h_0} P = 3 d_{h_0} d_J P. \tag{4.4}$$

Using formula [12, (4.8)], the corresponding horizontal projectors for the two sprays S and S_0 are related by

$$h = h_0 - PJ - d_J P \otimes \mathbb{C}. \tag{4.5}$$

We use that $\mathbb{C}(P^2 - S_0 P) = 2(P^2 - S_0 P)$ as well as the formulae (4.3) and (4.5) to obtain

$$d_h \rho = d_{h_0}(P^2 - S_0 P) - P d_J \rho - 2 \rho d_J P. \tag{4.6}$$

In Section 4.1 we will use the conditions of [13, Theorem 4.1] to test whether the spray S, given by formula (4.1), is metrizable by a Finsler function of constant flag curvature. In Section 4.2 we will use the conditions of Theorem 3.1 to test whether the spray S is metrizable by a Finsler function of scalar flag curvature. In each subsection, we show how to construct examples of sprays that are metrizable by such Finsler functions.

4.1. Solutions to Hilbert's fourth problem by Finsler functions of constant flag curvature. The projectively flat spray S, given by formula (4.1), is isotropic, the Ricci scalar, ρ, and the semi-basic 1-form α are given by formulae (4.3). According to [13, Theorem 4.1], the spray S is metrizable by a Finsler function of constant flag curvature if and only if the following three conditions are satisfied:

(C1) $d_J \alpha = 0$;
(C2) $d_h \rho = 0$;
(C3) $\operatorname{rank}(d d_J \rho) = 2n$.

We study now the first condition (C1). Since the spray S_0 is flat, it follows that $R_0 = [h_0, h_0]/2 = 0$ and therefore $d^2_{h_0} = 0$. Using a Poincaré-type lemma for the differential operator d_{h_0}, and formula (4.4), it follows that the condition (C1) is satisfied if and only if there exists a locally defined, 0-homogeneous, smooth function g on $\Omega \times \mathbb{R}^n \setminus \{0\}$ such that

$$d_J P = d_{h_0} g. \tag{4.7}$$

From the above formula, by applying the inner product i_{S_0} to both sides, we obtain

$$P = \mathbb{C}(P) = i_{S_0} d_J P = i_{S_0} d_{h_0} g = S_0(g). \tag{4.8}$$

In view of this formula, we obtain that the Ricci scalar, ρ, in formula (4.3), can be expressed as follows:

$$\rho = (S_0(g))^2 - S_0^2(g). \tag{4.9}$$

Using formula (4.6), as well as the above formulae, we obtain that the second condition (C2) is satisfied if and only if

$$d_{h_0}\rho - S_0(g) d_J\rho - 2\rho d_{h_0} g = 0.$$

We can write above formula, which is equivalent to the condition (C2), as follows

$$d_{h_0}(\exp(-2g)\rho) + \tfrac{1}{2} S_0(\exp(-2g)) d_J\rho = 0. \tag{4.10}$$

Remark. Each solution g of (4.10) determines a projectively flat Finsler metric $F^2 = |(S_0(g))^2 - S_0^2(g)|$, of constant flag curvature, if and only if the regularity condition (C3) is satisfied.

Next, we provide some examples of such functions g.

4.1.1. *Example.* Consider the open disk $\Omega = \{x \in \mathbb{R}^n, |x| < 1\}$, the function $g(x) = -\ln\sqrt{1 - |x|^2}$, and the projectively flat spray $S = S_0 - 2g^c\mathbb{C} \in \mathfrak{X}(\Omega \times \mathbb{R}^n)$. The particular form of the projective factor $P(x, y) = g^c(x, y) = S_0(g) = y^i \partial g/\partial x^i$ assures that the function g is a solution of (4.7), which means that the condition (C1) is satisfied.

For this spray S, the Ricci scalar given by formula (4.9) has the following expression

$$\rho(x, y) = -\frac{|y|^2(1 - |x|^2) + \langle x, y\rangle^2}{(1 - |x|^2)^2}. \tag{4.11}$$

Since the function g is a solution of (4.10) it follows that the condition (C2) is satisfied.

It remains to check the regularity condition (C3). By a direct computation we have $dd_J\rho = 2g_{ij}\delta y^i \wedge dx^j$, where

$$g_{ij} = \frac{\partial^2 g}{\partial x^i \partial x^j} - \frac{\partial g}{\partial x^i}\frac{\partial g}{\partial x^j} = \frac{1}{1 - |x|^2}\left(\delta_{ij} + \frac{x_i x_j}{1 - |x|^2}\right), \tag{4.12}$$

is the Klein metric on the unit ball, see [29, Example 11.3.1]. Therefore, we have that the projectively flat spray S is the geodesic spray of the Klein metric,

$$F^2(x, y) = -\rho(x, y) = \frac{|y|^2(1 - |x|^2) + \langle x, y\rangle^2}{(1 - |x|^2)^2}, \tag{4.13}$$

which has constant flag curvature $\kappa = \rho/F^2 = -1$.

4.1.2. *Example.* If we consider the function $g(x) = -\ln\sqrt{1+|x|^2}$, a solution of (4.10), we obtain that the spray $S = S_0 - 2g^c\mathbb{C} \in \mathfrak{X}(\mathbb{R}^n \times \mathbb{R}^n)$ is metrizable by the following metric on $\mathbb{R}^n$,

$$F^2 = S_0(g^c) - (g^c)^2 = \frac{|y|^2(1+|x|^2) - \langle x, y\rangle^2}{(1+|x|^2)^2}, \tag{4.14}$$

of constant curvature $\kappa = 1$ (see [29, Example 11.3.2]).

4.2. Solutions to Hilbert's fourth problem by Finsler functions of scalar flag curvature. In this subsection, we try to extend the question we addressed in the previous subsection, from constant flag curvature to scalar flag curvature. Therefore, we consider a domain $\Omega \subset \mathbb{R}^n$ and let $S_0 \in \mathfrak{X}(\Omega \times \mathbb{R}^n)$ be the flat spray. We will provide an example of a projective deformation $S = S_0 - 2P\mathbb{C}$, for a 1-homogeneous function $P \in C^\infty(\Omega \times \mathbb{R}^n)$, which will lead to a spray metrizable by a Finsler function of scalar flag curvature. Such a projectively flat Finsler function will therefore be a solution to Hilbert's fourth problem.

As we have seen already, the spray $S = S_0 - 2P\mathbb{C}$ is isotropic, the Ricci scalar, ρ, and the semi-basic 1-form α are given by formulae (4.3). Since the spray S is isotropic, according to Theorem 3.1, it follows that S is Finsler metrizable, which is equivalent to being metrizable by a Finsler function of scalar Flag curvature, if and only if the following three conditions are satisfied:

(S1) $d_J(\alpha/\rho) = 0$;
(S2) $D^h(\alpha/\rho) = 0$;
(S3) the regularity condition (iv) of Theorem 3.1.

Next we provide an example of a projective factor P, which has a very similar form with those considered in the previous two examples. However, for the function P in the next example, the projectively flat spray S satisfies the conditions (S1), (S2) and (S3) and hence will be metrizable by a Finsler function of scalar flag curvature.

4.2.1. *Example.* For the open disk $\Omega = \{x \in \mathbb{R}^n, |x| < 1\}$ in $\mathbb{R}^n$, we consider the function $g \in C^\infty(\Omega \times (\mathbb{R}^n \setminus \{0\}))$, $g(x, y) = \ln\sqrt{|y| + \langle x, y\rangle}$, and the projectively flat spray $S \in \mathfrak{X}(\Omega \times (\mathbb{R}^n \setminus \{0\}))$, given by

$$S = S_0 - 2S_0(g)\mathbb{C} = y^i\frac{\partial}{\partial x^i} - \frac{|y|^2 y^i}{|y| + \langle x, y\rangle}\frac{\partial}{\partial y^i}. \tag{4.15}$$

The projective factor $P = S_0(g)$ is given by

$$P(x, y) = \frac{1}{2}\frac{|y|^2}{|y| + \langle x, y\rangle}.$$

Using the first formula (4.3) we obtain that the Ricci scalar is given by

$$\rho(x, y) = 3P^2(x, y) = \frac{3}{4}\frac{|y|^4}{(|y| + \langle x, y\rangle)^2}. \tag{4.16}$$

Using the above formula for ρ and the second formula (4.3) we obtain that the semi-basic 1-form α is given by

$$\alpha = -3(d_{h_0}P + Pd_JP) = \frac{3|y|^2}{4(|y| + \langle x, y\rangle)^3}(y_i|y| + x_i|y|^2)\, dx^i. \tag{4.17}$$

Using the formulae (4.16) and (4.17) it follows that the semi-basic 1-form α/ρ is d_J-closed, since

$$\frac{\alpha}{\rho} = \frac{1}{|y| + \langle x, y\rangle}\left(\frac{y_i}{|y|} + x_i\right)dx^i = d_J f, f(x, y) = \ln(|y| + \langle x, y\rangle). \tag{4.18}$$

From the above formula we have that the first condition (S1) is satisfied. As we have shown in the proof of Theorem 3.1, the second condition (S2) is equivalent to the fact that $d_h f$ is a basic 1-form on Ω. Using formula (4.5) for the horizontal projector h and expression (4.18) for the function f, we have that $d_h f = 0$. The regularity condition (S3) is also satisfied and, hence, by formula (3.9), we obtain that

$$F(x, y) = \exp f(x, y) = |y| + \langle x, y\rangle, \tag{4.19}$$

is a Finsler function. The function F is a Finsler function of Numata type, see [5, Section 3.9.B]. The Finsler function F has scalar flag curvature, which is given by formula (3.12),

$$\kappa(x, y) = \frac{\rho}{F^2} = \frac{3}{4}\frac{|y|^4}{(|y| + \langle x, y\rangle)^4}. \tag{4.20}$$

The geodesics of the Finsler function F, given by formula (4.19), are segments of straight lines in the open disk Ω. As expected, from the recent result of Álvarez Paiva in [2], the nonreversible Finsler function F is the sum of a reversible projective metric and an exact 1-form.

5. Examples

In Example 4.2.1 we have studied a spray, in arbitrary dimension, which is metrizable by a Finsler function of scalar flag curvature. We have tested the Finsler metrizability of this spray using the SFC test of Theorem 3.1 and made use of the algorithm provided by the theorem to construct the corresponding Finsler function.

In this section we will use again the conditions of Theorem 3.1 to test whether or not some other examples of sprays are Finsler metrizable. We will also see that the regularity condition (iv) of Theorem 3.1 can be relaxed and we can search for sprays metrizable by conic pseudo-, or degenerate Finsler functions.

5.1. A spray metrizable by a conic pseudo-Finsler function. Consider the following affine spray on some domain $M \subset \mathbb{R}^2$, where two smooth functions ϕ and ψ are defined,

$$S = y^1\frac{\partial}{\partial x^1} + y^2\frac{\partial}{\partial x^2} - \phi(x^1, x^2)(y^1)^2\frac{\partial}{\partial y^1} - \psi(x^1, x^2)(y^2)^2\frac{\partial}{\partial y^2}.$$

Using formulae (2.4), the local components of the corresponding Jacobi endomorphism are given by

$$R^1_1 = -\phi_{x^2} y^1 y^2, \quad R^1_2 = \phi_{x^2} (y^1)^2, \quad R^2_1 = \psi_{x^1} (y^2)^2, \quad R^2_2 = -\psi_{x^1} y^1 y^2.$$

According to formula (2.6), the Ricci scalar is given by

$$\rho = R^1_1 + R^2_2 = -y^1 y^2 (\phi_{x^2} + \psi_{x^1}).$$

The case when $\phi_{x^2} = -\psi_{x^1} \neq 0$ has been studied in Example 8.2.4 from [29]. In this case, we have that the Ricci scalar is $\rho = 0$ while $\Phi \neq 0$ and, hence, S is not Finsler metrizable.

We now pay attention to the case $\rho \neq 0$. In this case, using the four conditions of Theorem 3.1, we will prove that S is Finsler metrizable if and only if there exists a constant $c \in \mathbb{R} \setminus \{0, 1\}$, such that

$$c\phi_{x^2} = (1 - c)\psi_{x^1}. \tag{5.1}$$

Since S is a spray on a two-dimensional manifold, it follows that it is isotropic and, hence, the first condition of Theorem 3.1 is satisfied. The two components of the semi-basic 1-form $\alpha = \alpha_1\, dx^1 + \alpha_2\, dx^2$, which appear in the expression (2.7) of the Jacobi endomorphism, are given by [13, (4.4)]:

$$\alpha_1 = \frac{R^2_2}{y^1} = -\psi_{x^1} y^2, \quad \alpha_2 = \frac{R^1_1}{y^2} = -\phi_{x^2} y^1.$$

The last three conditions of Theorem 3.1 refer to the semi-basic 1-form α/ρ, which is given by

$$\frac{\alpha}{\rho} = \frac{\psi_{x^1}}{(\phi_{x^2} + \psi_{x^1}) y^1}\, dx^1 + \frac{\phi_{x^2}}{(\phi_{x^2} + \psi_{x^1}) y^2}\, dx^2.$$

For the second condition of Theorem 3.1, one can immediately check that $d_J(\alpha/\rho) = 0$ and therefore there exists a function f defined on the conic region $A = \{(x^1, x^2, y^1, y^2) \in TM, y^1 > 0, y^2 > 0\}$ of T_0M, such that $\alpha/\rho = d_J f$. The function f is given by

$$f(x, y) = \frac{1}{\phi_{x^2} + \psi_{x^1}} (\psi_{x_1} \ln y^1 + \phi_{x_2} \ln y^2). \tag{5.2}$$

For the third condition of Theorem 3.1, we have to test whether $d_h f$ is a basic 1-form. For the spray S, the local coefficients N^i_j, of the nonlinear connection are given by

$$N^1_1 = \phi y^1, \quad N^1_2 = N^2_1 = 0, \quad N^2_2 = \psi y^2.$$

It follows that

$$d_h f = \frac{\delta f}{\delta x^1} dx^1 + \frac{\delta f}{\delta x^2} dx^2, \quad \frac{\delta f}{\delta x^1} = \frac{\partial f}{\partial x^1} - \frac{\phi \psi_{x^1}}{\phi_{x^2} + \psi_{x^1}}, \quad \frac{\delta f}{\delta x^2} = \frac{\partial f}{\partial x^2} - \frac{\psi \phi_{x^2}}{\phi_{x^2} + \psi_{x^1}}.$$

Therefore, $d_h f$ is a basic 1-form if and only if there exist two real constant c_1 and c_2 such that

$$\frac{\psi_{x^1}}{\phi_{x^2}+\psi_{x^1}} = c_1, \qquad \frac{\phi_{x^2}}{\phi_{x^2}+\psi_{x^1}} = c_2. \tag{5.3}$$

Expression (5.2) and the condition $\mathbb{C}(f) = 1$ implies $c_1 + c_2 = 1$. Formula (5.3) is equivalent to formula (5.1), for $c = c_1$ and $c_2 = 1 - c$.

We will show that, within the given assumptions (5.1), the last condition of Theorem 3.1 is satisfied. We have that

$$\frac{\alpha}{\rho} = \frac{c}{y^1}\,dx^1 + \frac{1-c}{y^2}\,dx^2$$

and, therefore,

$$d\left(\frac{\alpha}{\rho}\right) + 2i_{\mathbb{F}}\frac{\alpha}{\rho}\wedge\frac{\alpha}{\rho} = \frac{c(2c-1)}{(y^1)^2}\delta y^1\wedge dx^1 + \frac{c(2-2c)}{y^1y^2}(\delta y^1\wedge dx^2 + \delta y^2\wedge dx^1) + \frac{(1-c)(1-2c)}{(y^2)^2}\delta y^2\wedge dx^2,$$

which is nondegenerate and hence it is a symplectic form on $A\subset T_0M$.

We have shown that the spray S is Finsler metrizable if and only if the condition (5.1) is satisfied. We will show now how we can construct the Finsler function that metricizes the spray. To simplify the calculations, we choose the constant $c = 1/2$ and the functions $\phi(x^1,x^2) = \psi(x^1,x^2) = 2g'(x^1+x^2)/g(x^1+x^2)$, where $g(t)$ is a nonvanishing smooth function. In this case, one can see that the condition (5.1) is satisfied.

For this choice we have that the basic 1-form $d_h f$ is given by

$$d_h f = -\frac{g'}{g}\,dx^1 - \frac{g'}{g}\,dx^2 = da, \quad a(x^1,x^2) = -\ln g(x^1+x^2).$$

According to formula (3.9), it follows that

$$F(x,y) = \exp(f-a) = \frac{\sqrt{y^1y^2}}{g(x^1+x^2)},$$

metricizes the spray S for the given choice of the functions ϕ and ψ. The scalar flag curvature is given by formula (3.12), and for the above Finsler function is

$$\kappa = \frac{\rho}{F^2} = 4(g''g - (g')^2).$$

For the particular case, when $g(t) = t/2$ we obtain the case of constant sectional curvature $\kappa = -1$ studied in [13, Section 5.4].

5.2. A spray metrizable by a degenerate Finsler function. We present now an example of a spray that is metrizable by a degenerate Finsler function of scalar flag curvature. This means that the first three conditions of Theorem 3.1 are satisfied, while the last one it is not. On $M = \mathbb{R}^2$, consider the following system of SODEs:

$$\frac{d^2x^1}{dt^2} + 2\frac{dx^1}{dt}\frac{dx^2}{dt} = 0, \qquad \frac{d^2x^2}{dt^2} - \left(\frac{dx^2}{dt}\right)^2 = 0. \tag{5.4}$$

The corresponding spray $S \in \mathfrak{X}(TM)$ is given by

$$S = y^1\frac{\partial}{\partial x^1} + y^2\frac{\partial}{\partial x^2} - 2y^1y^2\frac{\partial}{\partial y^1} + (y^2)^2\frac{\partial}{\partial y^2}.$$

Using the formulae (2.4) and (2.6), the local components of the corresponding Jacobi endomorphism and the Ricci scalar are given by

$$R^1_1 = -2(y^2)^2, \quad R^2_2 = 0, \quad \rho = -2(y^2)^2.$$

Since S is a two-dimensional spray, it follows that it is isotropic and, hence, first condition of Theorem 3.1 is satisfied. The semi-basic 1-form $\alpha/\rho = \alpha_1/\rho dx^2 + \alpha_2/\rho dx^2$ has the components:

$$\frac{\alpha_1}{\rho} = \frac{R^2_2}{y^1\rho} = 0, \qquad \frac{\alpha_2}{\rho} = \frac{R^1_1}{y^2\rho} = \frac{1}{y^2}.$$

From the above formulae, one can immediately check that $d_J(\alpha/\rho) = 0$ and, hence, the second condition of Theorem 3.1 is satisfied. Moreover, we have that there exists a function $f \in C^\infty(T_0M)$ such that

$$\frac{\alpha}{\rho} = d_Jf, \quad \text{for } f(x,y) = \ln|y^2|.$$

Third condition of Theorem 3.1 is satisfied if and only if d_hf is a basic 1-form. By direct calculation we have that this is true, since $d_hf = dx^2$. More than that, for $a(x^1, x^2) = x^2$, we have that $d_hf = da$. Therefore, the function

$$F(x,y) = \exp(f(x,y) - a(x)) = \exp(-x^2)y^2$$

is a degenerate Finsler function that metricizes the given system (5.4). This degenerate Finsler function has scalar flag curvature, given by formula (3.12), which in our case is

$$\kappa = \frac{\rho}{F^2} = \frac{-2}{\exp(-x^2)}.$$

It can be directly checked that any solution of the system (5.4) is also a solution of the Euler–Lagrange equations for F^2. Some other nonhomogeneous Lagrangian functions that metricizes the system (5.4) where determined in [3, Example 7.10].

5.3. A spray that is not Finsler metrizable. We consider now an example of a spray that is not Finsler metrizable, and this is due to the fact that the third condition of Theorem 3.1 is not satisfied. On $M = \mathbb{R}^2$, we consider the following system of SODEs:

$$\frac{d^2x^1}{dt^2} + \left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 = 0, \quad \frac{d^2x^2}{dt^2} + 4\frac{dx^1}{dt}\frac{dx^2}{dt} = 0. \tag{5.5}$$

The above system can be identified with a spray $S \in \mathfrak{X}(TM)$, which is given by

$$S = y^1\frac{\partial}{\partial x^1} + y^2\frac{\partial}{\partial x^2} - ((y^1)^2 + (y^2)^2)\frac{\partial}{\partial y^1} - 4y^1y^2\frac{\partial}{\partial y^2}.$$

We make use of formulae (2.4) and (2.6) to compute the local components of the corresponding Jacobi endomorphism and the Ricci scalar, which are given by

$$R^1_1 = -(y^2)^2, \quad R^2_2 = -2(y^1)^2, \quad \rho = -2(y^1)^2 - (y^2)^2.$$

Again, the spray S is two-dimensional and, hence, it is isotropic, which means that the first condition of Theorem 3.1 is satisfied. The other conditions refer to the semi-basic 1-form $\alpha/\rho = (\alpha_1/\rho)\,dx^2 + (\alpha_2/\rho)\,dx^2$, whose components are given by

$$\frac{\alpha_1}{\rho} = \frac{R^2_2}{y^1\rho} = \frac{2y^1}{2(y^1)^2 + (y^2)^2}, \quad \frac{\alpha_2}{\rho} = \frac{R^1_1}{y^2\rho} = \frac{y^2}{2(y^1)^2 + (y^2)^2}.$$

From the above formulae, it follows that $d_J(\alpha/\rho) = 0$, which means that the second condition of Theorem 3.1 is satisfied. Therefore, there exists a function $f \in C^\infty(T_0M)$ such that

$$\frac{\alpha}{\rho} = d_J f, \quad \text{for } f(x, y) = \ln(2(y^1)^2 + (y^2)^2).$$

For the above considered function f we can check that $d_h f$ is not a basic 1-form. It follows then that third condition of Theorem 3.1 is not satisfied and consequently the spray is not Finsler metrizable. The system (5.5) has been borrowed from [3, Example 7.2], where it has been shown, using different techniques, that this system is not metrizable.

Acknowledgement

We appreciate the comments of the referee, especially those related to Douglas' work.

References

[1] J. C. Álvarez Paiva, 'Symplectic geometry and Hilbert's fourth problem', *J. Differential Geom.* **69** (2005), 353–378.
[2] J. C. Álvarez Paiva, 2013 Asymmetry in Hilbert's fourth problem. arXiv:1301.2524.
[3] I. Anderson and G. Thompson, 'The inverse problem of the calculus of variations for ordinary differential equations', *Mem. Amer. Math. Soc.* **98** (1992), 1–110.

[4] P. L. Antonelli, R. S. Ingarden and M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology* (Kluwer, Dordrecht, 1993).
[5] D. Bao, S.-S. Chern and Z. Shen, *An introduction to Riemann–Finsler geometry* (Springer, New York, 2000).
[6] D. Bao and C. Robles, 'Ricci and flag curvatures in Finsler geometry', in: *A Sampler of Riemann–Finsler Geometry*, Mathematical Sciences Research Institute Publications, 50 (ed. D. Bao) (Cambridge University Press, Cambridge, 2004), 197–259.
[7] L. Berwald, 'On Finsler and Cartan geometries. III: two-dimensional Finsler spaces with rectilinear extremals', *Ann. Math.* **42** (1941), 84–112.
[8] R. L. Bryant, 'Some remarks on Finsler manifolds with constant flag curvature', *Houston J. Math.* **28** (2002), 221–262.
[9] I. Bucataru, O. Constantinescu and M. F. Dahl, 'A geometric setting for systems of ordinary differential equations', *Int. J. Geom. Methods Mod. Phys.* **8** (2011), 1291–1327.
[10] I. Bucataru and M. F. Dahl, 'Semi-basic 1-forms and Helmholtz conditions for the inverse problem of the calculus of variations', *J. Geom. Mech.* **1** (2009), 159–180.
[11] I. Bucataru and Z. Muzsnay, 'Projective metrizability and formal integrability', *SIGMA* **7** (2011), 114, 22 pages.
[12] I. Bucataru and Z. Muzsnay, 'Projective and Finsler metrizability: parametrization rigidity of geodesics', *Int. J. Math.* **23** (2012), 1250099.
[13] I. Bucataru and Z. Muzsnay, 'Sprays metrizable by Finsler functions of constant flag curvature', *Differential Geom. Appl.* **31** (2013), 405–415.
[14] S. Casey, M. Dunajski and P. Tod, 'Twistor geometry of a pair of second order ODEs', *Comm. Math. Phys.* **321** (2013), 681–701.
[15] M. Crampin, 'Isotropic and R-flat sprays', *Houston J. Math.* **33** (2007), 451–459.
[16] M. Crampin, 'Some remarks on the Finslerian version of Hilbert's fourth problem', *Houston J. Math.* **37** (2011), 369–391.
[17] M. Crampin, T. Mestdag and D. J. Saunders, 'Hilbert forms for a Finsler metrizable projective class of sprays', *Differential Geom. Appl.* **31** (2013), 63–79.
[18] M. Crampin, W. Sarlet, E. Martínez, G. B. Byrnes and G. E. Prince, 'Towards a geometrical understanding of Douglas' solution of the inverse problem of the calculus of variations', *Inv. Prob.* **10** (1994), 245–260.
[19] J. Douglas, 'Solution to the inverse problem of the calculus of variations', *Trans. Amer. Math. Soc.* **50** (1941), 71–128.
[20] J. Grifone, 'Structure presque-tangente et connexions. I', *Ann. Inst. Henri Poincare* **22** (1972), 287–334.
[21] J. Grifone and Z. Muzsnay, *Variational Principles for Second-order Differential Equations* (World Scientific, Singapore, 2000).
[22] M. A. Javaloyes and M. Sánchez, 'On the definition and examples of Finsler metrics', *Ann. Sc. Norm. Sup. Pisa.* doi:10.2422/2036-2145.201203-002, arXiv:1111.5066.
[23] I. Kolar, P. W. Michor and J. Slovak, *Natural Operations in Differential Geometry* (Springer-Verlag, Berlin, 1993).
[24] D. Krupka and A. E. Sattarov, 'The inverse problem of the calculus of variations for Finsler structures', *Math. Slovaca* **35** (1985), 217–222.
[25] O. Krupková, *The Geometry of Ordinary Variational Equations*, Lecture Notes in Mathematics 1678 (Springer-Verlag, Berlin, 1997).
[26] G. Morandi, C. Ferrario, G. Lo Vecchio, G. Marmo and C. Rubano, 'The inverse problem in the calculus of variations and the geometry of the tangent bundle', *Phys. Rep.* **188** (1990), 147–284.
[27] Z. Muzsnay, 'The Euler–Lagrange PDE and Finsler metrizability', *Houston J. Math.* **32** (2006), 79–98.
[28] W. Sarlet, G. Thompson and G. E. Prince, 'The inverse problem of the calculus of variations: the use of geometrical calculus in Douglas's analysis', *Trans. Amer. Math. Soc.* **354** (2002), 2897–2919.

[29] Z. Shen, *Differential Geometry of Spray and Finsler Spaces* (Springer, New York, 2001).
[30] Z. Shen, 'Projectively flat Finsler metrics of constant flag curvature', *Trans. Amer. Math. Soc.* **355** (2003), 1713–1728.
[31] J. Szilasi, 'A setting for spray and Finsler geometry', in: *Handbook of Finsler Geometry*, 2 (ed. P. L. Antonelli) (Kluwer, Dordrecht, 2003), 1183–1426.
[32] J. Szilasi and S. Vattamány, 'On the Finsler-metrizabilities of spray manifolds', *Period. Math. Hungar.* **44** (2002), 81–100.

IOAN BUCATARU, Faculty of Mathematics,
Alexandru Ioan Cuza University, Iaşi, Romania
e-mail: bucataru@uaic.ro

ZOLTÁN MUZSNAY, Institute of Mathematics,
University of Debrecen, Debrecen, Hungary
e-mail: muzsnay@science.unideb.hu