Finsler 2-manifolds with maximal holonomy group of infinite dimension

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Abstract

In this paper we are investigating the holonomy structure of Finsler 2-manifolds. We show that the topological closure of the holonomy group of a certain class of projectively flat Finsler 2-manifolds of constant curvature is maximal, that is isomorphic to the connected component of the diffeomorphism group of the circle. This class of 2-manifolds contains the standard Funk plane of constant negative curvature and the Bryant-Shen-spheres of constant positive curvature. The result provides the first examples describing completely infinite dimensional Finslerian holonomy structures.

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1. Introduction

The notion of the holonomy group of a Riemannian or Finslerian manifold can be introduced in a very natural way: it is the group generated by parallel translations along loops with respect to the associated linear, respectively homogeneous (nonlinear) connection. In contrast to the Finslerian case, the Riemannian holonomy groups have been extensively studied. One of the earliest fundamental results is the theorem of Borel and Lichnerowicz [1] from 1952, claiming that the holonomy group of a simply connected Riemannian manifold is a closed Lie subgroup of the orthogonal group $O(n)$. By now, the complete classification of Riemannian holonomy groups is known.

The holonomy properties of Finsler spaces is, however, essentially different from the Riemannian one, and it is far from being well understood. Compared to the Riemannian case, only few results are known. In [12] it was proved that the holonomy group of a Finsler manifold of nonzero constant curvature with dimension greater than 2 is not a compact Lie group. In [14] it has been shown that there exist large families of projectively flat Finsler manifolds of constant curvature such that their holonomy groups are not finite dimensional Lie groups. In [15] we

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characterized the projective Finsler manifolds of constant curvature having infinite dimensional holonomy group. The proofs in the above mentioned papers give estimates for the dimension of tangent Lie algebras of the holonomy group and therefore they do not give direct information about the infinite dimensional structure of the holonomy group.

Until now, perhaps because of technical difficulties, not a single infinite dimensional Finsler holonomy group has been determined. In this paper we provide the first such a description: we show that the topological closure of the holonomy group of a certain class of simply connected, projectively flat Finsler 2-manifolds of constant curvature is $\mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$, the connected component of the full diffeomorphism group of the circle. This class of Finsler 2-manifolds contains the positively complete standard Funk plane of constant negative curvature (positively complete standard Funk plane), and the complete irreversible Bryant-Shen-spheres of constant positive curvature ([18], [3]). We remark that for every simply connected Finsler 2-manifold the topological closure of the holonomy group is a subgroup of $\mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$. Consequently, in the examples mentioned above, the closed holonomy group is maximal. In the proof we use the constructive method developed in [14] to study the Lie algebras of vector fields on the indicatrix which are tangent to the holonomy group.

2. Preliminaries

Throughout this article, M is a C^∞ smooth manifold, $\mathfrak{X}^\infty(M)$ is the vector space of smooth vector fields on M and $\mathsf{Diff}^\infty(M)$ is the group of all C^∞-diffeomorphism of M. The first and the second tangent bundles of M are denoted by (TM, π, M) and (TTM, τ, TM), respectively.

A *Finsler manifold* is a pair $(M, \mathcal{F})$, where the norm function $\mathcal{F}\colon TM \to \mathbb{R}_+$ is continuous, smooth on $\hat{T}M := TM \setminus \{0\}$, its restriction $\mathcal{F}_x = \mathcal{F}|_{T_xM}$ is a positively homogeneous function of degree one and the symmetric bilinear form

$$g_{x,y}\colon (u,v) \mapsto g_{ij}(x,y)u^iv^j = \frac{1}{2}\frac{\partial^2\mathcal{F}_x^2(y+su+tv)}{\partial s\,\partial t}\Big|_{t=s=0}$$

is positive definite at every $y \in \hat{T}_xM$.

Geodesics of $(M, \mathcal{F})$ are determined by a system of 2nd order ordinary differential equation $\ddot{x}^i + 2G^i(x,\dot{x}) = 0$, $i = 1, \dots, n$ in a local coordinate system (x^i, y^i) of TM, where $G^i(x,y)$ are given by

$$G^i(x,y) := \frac{1}{4}g^{il}(x,y)\Big(2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\Big)y^jy^k. \tag{1}$$

A vector field $X(t) = X^i(t)\frac{\partial}{\partial x^i}$ along a curve $c(t)$ is said to be parallel with respect to the associated *homogeneous (nonlinear) connection* if it satisfies

$$D_{\dot{c}}X(t) := \Big(\frac{dX^i(t)}{dt} + G^i_j(c(t), X(t))\dot{c}^j(t)\Big)\frac{\partial}{\partial x^i} = 0, \tag{2}$$

where $G^i_j = \frac{\partial G^i}{\partial y^j}$.

The *horizontal Berwald covariant derivative* $\nabla_X\xi$ of $\xi(x,y) = \xi^i(x,y)\frac{\partial}{\partial y^i}$ by the vector field $X(x) = X^i(x)\frac{\partial}{\partial x^i}$ is expressed locally by

$$\nabla_X\xi = \left(\frac{\partial\xi^i(x,y)}{\partial x^j} - G^k_j(x,y)\frac{\partial\xi^i(x,y)}{\partial y^k} + G^i_{jk}(x,y)\xi^k(x,y)\right)X^j\frac{\partial}{\partial y^i}, \tag{3}$$

where we denote $G^i_{jk}(x,y) := \frac{\partial G^i_j(x,y)}{\partial y^k}$.

The *Riemannian curvature tensor* field $R = R^i_{jk}(x,y)dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^i}$ has the expression

$$R^i_{jk}(x,y) = \frac{\partial G^i_j(x,y)}{\partial x^k} - \frac{\partial G^i_k(x,y)}{\partial x^j} + G^m_j(x,y)G^i_{km}(x,y) - G^m_k(x,y)G^i_{jm}(x,y).$$

The manifold has *constant flag curvature* $\lambda \in \mathbb{R}$, if for any $x \in M$ the local expression of the Riemannian curvature is

$$R^i_{jk}(x,y) = \lambda(\delta^i_k g_{jm}(x,y)y^m - \delta^i_j g_{km}(x,y)y^m).$$

Assume that the Finsler manifold $(M, \mathcal{F})$ is locally projectively flat. Then for every point $x \in M$ there exists an *adapted* local coordinate system, that is a mapping $(x^1, \dots, x^n)$ on a neighbourhood U of x into the Euclidean space $\mathbb{R}^n$, such that the straight lines of $\mathbb{R}^n$ correspond to the geodesics of $(M, \mathcal{F})$. Then the *geodesic coefficients* are of the form

$$G^i = \mathcal{P} y^i, \quad G^i_k = \frac{\partial \mathcal{P}}{\partial y^k} y^i + \mathcal{P}\delta^i_k, \quad G^i_{kl} = \frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^l} y^i + \frac{\partial \mathcal{P}}{\partial y^k}\delta^i_l + \frac{\partial \mathcal{P}}{\partial y^l}\delta^i_k \tag{4}$$

where $\mathcal{P}(x,y)$ is a 1-homogeneous function in y, called the *projective factor* of $(M, \mathcal{F})$. According to Lemma 8.2.1 in [4] p.155, if $(M \subset \mathbb{R}^n, \mathcal{F})$ is a projectively flat manifold, then its projective factor can be computed using the formula

$$\mathcal{P}(x,y) = \frac{1}{2\mathcal{F}} \frac{\partial \mathcal{F}}{\partial x^i} y^i. \tag{5}$$

Example 1. (P. Funk, [5, 6, 7]) The *standard Funk manifold* $(\mathbb{D}^n, \mathcal{F})$ defined by the metric function

$$\mathcal{F}(x,y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2} \pm \frac{\langle x,y\rangle}{1-|x|^2} \tag{6}$$

on the unit disk $\mathbb{D}^n \subset \mathbb{R}^n$ is projectively flat with constant flag curvature $\lambda = -\frac{1}{4}$. Its projective factor can be computed using formula (5):

$$\mathcal{P}(x,y) = \frac{1}{2} \, \frac{\pm\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} + \langle x,y\rangle}{1-|x|^2}. \tag{7}$$

We call the standard Funk 2-manifold the *standard Funk plane*.

Example 2. The *Bryant-Shen sphere*s $(\mathbb{S}^n, \mathcal{F}_\alpha)_{|\alpha|<\frac{\pi}{2}}$, are the elements of a 1-parameter family of projectively flat complete Finsler manifolds with constant flag curvature $\lambda = 1$ defined on the n-sphere $\mathbb{S}^n$. The metric function and the projective factor at $0 \in \mathbb{R}^n$ have the form

$$\mathcal{F}(0,y) = |y| \cos\alpha, \quad \mathcal{P}(0,y) = |y| \sin\alpha, \quad \text{with} \quad |\alpha| < \frac{\pi}{2} \tag{8}$$

in a local coordinate system corresponding to the Euclidean canonical coordinates, centered at $0 \in \mathbb{R}^n$. R. Bryant in [2], [3] introduced and studied this class of Finsler metrics on $\mathbb{S}^2$ where great circles are geodesics. Z. Shen generalized its construction to $\mathbb{S}^n$ and obtained the expression (8) (cf. Example 7.1. in [18] and Example 8.2.9 in [4]).

3. Holonomy group as a subgroup of the diffeomorphism group of the indicatrix

The group $\mathsf{Diff}^\infty(K)$ of diffeomorphisms of a compact manifold K is an infinite dimensional Lie group belonging to the class of Fréchet Lie groups. The Lie algebra of $\mathsf{Diff}^\infty(K)$ is the Lie algebra $\mathfrak{X}^\infty(K)$ of smooth vector fields on K endowed with the negative of the usual Lie bracket of vector fields. $\mathsf{Diff}^\infty(K)$ is modeled on the locally convex topological Fréchet vector space $\mathfrak{X}^\infty(K)$. A sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathfrak{X}^\infty(K)$ converges to f in the topology of $\mathfrak{X}^\infty(K)$ if and only if the functions f_j and all their derivatives converge uniformly to f, respectively to the corresponding derivatives of f. We note that the difficulty of the theory of Fréchet manifolds comes from the fact that the inverse function theorem and the existence theorems for differential equations, which are well known for Banach manifolds, are not true in this category. These problems have led to the concept of regular Fréchet Lie groups (cf. H. Omori [16] Chapter III, A. Kriegl – P. W. Michor [11] Chapter VIII). The distinguishing properties of regular Fréchet Lie groups can be summarized as the existence of smooth exponential map from the Lie algebra of the Fréchet Lie groups to the group itself, and the existence of product integrals, which produces the convergence of some approximation methods for solving differential equations (cf. Section III.5. in [16], pp. 83 –89). In particular $\mathsf{Diff}^\infty(K)$ is a topological group which is an inverse limit of Lie groups modeled on Banach spaces and hence it is a regular Fréchet Lie group (Corollary 5.4 in [16]).

Let H be a subgroup of the diffeomorphism group $\mathsf{Diff}^\infty(K)$ of a differentiable manifold K. A vector field $X \in \mathfrak{X}^\infty(K)$ is called *tangent to* $H \subset \mathsf{Diff}^\infty(K)$ if there exists a C^1-differentiable 1-parameter family $\{\Phi(t) \in H\}_{t\in\mathbb{R}}$ of diffeomorphisms of K such that $\Phi(0) = \mathsf{Id}$ and $\frac{d\Phi(t)}{dt}\Big|_{t=0} = X$. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{X}^\infty(K)$ is called *tangent to* H, if all elements of $\mathfrak{h}$ are tangent vector fields to H.

We denote by $(\mathcal{I}M, \pi, M)$ the *indicatrix bundle* of the Finsler manifold $(M, \mathcal{F})$, the *indicatrix* $\mathcal{I}_xM$ at $x \in M$ is the compact hypersurface $\mathcal{I}_xM := \{y \in T_xM;\ \mathcal{F}(y) = 1\}$ in T_xM which is diffeomorphic to the sphere $\mathbb{S}^{n-1}$, if $\dim(M) = n$. The homogeneous (nonlinear) parallel translation $\tau_c : T_{c(0)}M \to T_{c(1)}M$ along a curve $c : [0,1] \to M$ preserves the value of the Finsler function, hence it induces a map $\tau_c\colon \mathcal{I}_{c(0)}M \longrightarrow \mathcal{I}_{c(1)}M$ between the indicatrices.

The *holonomy group* $\mathsf{Hol}_x(M)$ of the Finsler manifold $(M, \mathcal{F})$ at a point $x \in M$ is the subgroup of the group of diffeomorphisms $\mathsf{Diff}^\infty(\mathcal{I}_xM)$ generated by homogeneous (nonlinear) parallel translations of $\mathcal{I}_xM$ along piece-wise differentiable closed curves initiated at the point $x \in M$. The *closed holonomy group* is the topological closure $\overline{\mathsf{Hol}_x(M)}$ of the holonomy group with respect of the Fréchet topology of $\mathsf{Diff}^\infty(\mathcal{I}_xM)$.
We remark that the diffeomorphism group $\mathsf{Diff}^\infty(\mathcal{I}_xM)$ of the indicatrix $\mathcal{I}_xM$ is a regular infinite dimensional Lie group modeled on the vector space $\mathfrak{X}^\infty(\mathcal{I}_xM)$. In this category the group structure is locally determined by the Lie algebra $\mathfrak{X}^\infty(\mathcal{I}_xM)$ of the Lie group $\mathsf{Diff}^\infty(\mathcal{I}_xM)$ (cf. [11, 16]).

For any vector fields $X, Y \in \mathfrak{X}^\infty(M)$ on M the vector field $\xi = R(X,Y) \in \mathfrak{X}^\infty(\mathcal{I}M)$ is called a *curvature vector field* of $(M, \mathcal{F})$ (see [12]). The Lie algebra $\mathfrak{R}(M)$ of vector fields generated by the curvature vector fields of $(M, \mathcal{F})$ is called the *curvature algebra* of $(M, \mathcal{F})$. The restriction $\mathfrak{R}_x(M) := \{\, \xi\big|_{\mathcal{I}_xM}\,;\ \xi \in \mathfrak{R}(M)\,\} \subset \mathfrak{X}^\infty(\mathcal{I}_xM)$ of the curvature algebra to an indicatrix $\mathcal{I}_xM$ is called the *curvature algebra at the point* $x \in M$.

The *infinitesimal holonomy algebra* of $(M, \mathcal{F})$ is the smallest Lie algebra $\mathfrak{hol}^*(M)$ of vector fields on the indicatrix bundle $\mathcal{I}M$ satisfying the following properties

a) any curvature vector field ξ belongs to $\mathfrak{hol}^*(M)$,

b) if $\xi \in \mathfrak{hol}^*(M)$ and $X \in \mathfrak{X}^\infty(M)$ then the horizontal Berwald covariant derivative $\nabla_X \xi$ belongs to $\mathfrak{hol}^*(M)$.

The restriction $\mathfrak{hol}^*_x(M) := \{\xi\big|_{\mathcal{I}_x M} \ ; \ \xi \in \mathfrak{hol}^*(M)\} \subset \mathfrak{X}^\infty(\mathcal{I}_x M)$ of the infinitesimal holonomy algebra to an indicatrix $\mathcal{I}_x M$ is called the *infinitesimal holonomy algebra at the point* $x \in M$. One has $\Re(M) \subset \mathfrak{hol}^*(M)$ and $\Re_x(M) \subset \mathfrak{hol}^*_x(M)$ for any $x \in M$ (see [13]).

Roughly speaking, the image of the curvature tensor (the curvature vector fields) determines the curvature algebra, which generates (with the bracket operation and the covariant derivation) the infinitesimal holonomy algebra. Localising these object at a point $x \in M$ we obtain the curvature algebra and the infinitesimal holonomy algebra at $x \in M$.

The following assertion will be an important tool in the next discussion:

The infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ *at any point* $x \in M$ *is tangent to the holonomy group* $\mathsf{Hol}_x(M)$. (Theorem 6.3 in [13]).

The holonomy group and its topological closure are interesting geometrical object which reflects geometric properties of the Finsler manifold. In the characterization of the closed holonomy group we use the following

Proposition 3.1. *The group generated by the exponential image of the infinitesimal holonomy algebra* $\mathfrak{hol}^*_x(M)$ *at a point* $x \in M$ *with respect to the exponential map* $\exp : \mathfrak{X}^\infty(\mathcal{I}_x M) \to \mathsf{Diff}^\infty(\mathcal{I}_x M)$ *is a subgroup of the closed holonomy group* $\overline{\mathsf{Hol}_x(M)}$.

Proof. Let us denote by $\langle \exp(\mathfrak{hol}^*_x(M))\rangle$ the group generated by the exponential image of $\mathfrak{hol}^*_x(M)$. For any element $X \in \mathfrak{hol}^*_x(M)$ there exists a C^1-differentiable 1-parameter family $\{\Phi(t) \in \mathsf{Hol}_x(M)\}_{t\in\mathbb{R}}$ of diffeomorphisms of the indicatrix $\mathcal{I}_x M$ such that $\Phi(0) = \mathsf{Id}$ and $\frac{d\Phi}{dt}\big|_{t=0} = X$. Then, considering $\Phi(t)$ as "hair" and using the argument of Corollary 5.4. in [16], p. 85, we get that $\Phi^n(\frac{t}{n}) = \Phi(\frac{t}{n}) \circ \cdots \circ \Phi(\frac{t}{n})$ in $\mathsf{Hol}_x(M)$, as a sequence of $\mathsf{Diff}^\infty(\mathcal{I}_x M)$, converges uniformly in all derivatives to $\exp(tX)$. It follows that we have

$$\{\exp(tX); t \in \mathbb{R}\} \subset \overline{\mathsf{Hol}_x(M)}$$

for any $X \in \mathfrak{hol}^*_x(M)$ and therefore $\exp(\mathfrak{hol}^*_x(M)) \subset \overline{\mathsf{Hol}_x(M)}$. Naturally, if we consider the generated group, denoted by $\langle \exp(\mathfrak{hol}^*_x(M))\rangle$, then the relation is preserved, that is

$$\langle \exp(\mathfrak{hol}^*_x(M))\rangle \subset \overline{\mathsf{Hol}_x(M)},$$

which proves the proposition.

4. The group $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$ and the Fourier algebra

Let $(M, \mathcal{F})$ be a Finsler 2-manifold. In this case the indicatrix is diffeomorphic to the unit circle $\mathbb{S}^1$, at any point $x \in M$. Moreover, if there exists a non-vanishing curvature vector field at $x \in M$ then any other curvature vector field at $x \in M$ is proportional to it, which means that the curvature algebra is at most 1-dimensional. The infinitesimal holonomy algebra however, can be an infinite dimensional subalgebra of $\mathfrak{X}^\infty(\mathbb{S}^1)$, therefore the holonomy group can be an infinite dimensional subgroup of $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$, cf. [14].

Let $\mathbb{S}^1 = \mathbb{R} \mod 2\pi$ be the unit circle with the standard counterclockwise orientation. The group $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$ of orientation preserving diffeomorphisms of the $\mathbb{S}^1$ is the connected component of $\mathsf{Diff}^\infty(\mathbb{S}^1)$. The Lie algebra of $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$ is the Lie algebra $\mathfrak{X}^\infty(\mathbb{S}^1)$ – denoted also by $\mathsf{Vect}(\mathbb{S}^1)$ in the literature – can be written in the form $f(t)\frac{d}{dt}$, where f is a 2π-periodic smooth functions on the real line $\mathbb{R}$. A sequence $\{f_j\frac{d}{dt}\}_{j\in\mathbb{N}} \subset \mathsf{Vect}(\mathbb{S}^1)$ converges to $f\frac{d}{dt}$ in the Fréchet topology of $\mathsf{Vect}(\mathbb{S}^1)$ if and only if the functions f_j and all their derivatives converge uniformly to f, respectively to the corresponding derivatives of f. The Lie bracket on $\mathsf{Vect}(\mathbb{S}^1)$ is given by

$$\Big[f\frac{d}{dt}, g\frac{d}{dt}\Big] = \Big(g\frac{df}{dt} - \frac{dg}{dt}f\Big)\frac{d}{dt}.$$

The *Fourier algebra* $\mathsf{F}(\mathbb{S}^1)$ on $\mathbb{S}^1$ is the Lie subalgebra of $\mathsf{Vect}(\mathbb{S}^1)$ consisting of vector fields $f\frac{d}{dt}$ such that $f(t)$ has finite Fourier series, i.e. $f(t)$ is a Fourier polynomial. The vector fields $\{\frac{d}{dt}, \cos nt\frac{d}{dt}, \sin nt\frac{d}{dt}\}_{n\in\mathbb{N}}$ provide a basis for $\mathsf{F}(\mathbb{S}^1)$. A direct computation shows that the vector fields

$$\frac{d}{dt}, \quad \cos t\frac{d}{dt}, \quad \sin t\frac{d}{dt}, \quad \cos 2t\frac{d}{dt}, \quad \sin 2t\frac{d}{dt} \tag{9}$$

generate the Lie algebra $\mathsf{F}(\mathbb{S}^1)$. The complexification $\mathsf{F}(\mathbb{S}^1)\otimes_{\mathbb{R}}\mathbb{C}$ of $\mathsf{F}(\mathbb{S}^1)$ is called the *Witt algebra* $\mathsf{W}(\mathbb{S}^1)$ on $\mathbb{S}^1$ having the natural basis $\{ie^{int}\frac{d}{dt}\}_{n\in\mathbb{Z}}$, with the Lie bracket $[ie^{imt}\frac{d}{dt}, ie^{int}\frac{d}{dt}] = i(m-n)e^{i(n-m)t}\frac{d}{dt}$.

Lemma 4.1. *The group* $\left\langle \overline{\exp(\mathsf{F}(\mathbb{S}^1))} \right\rangle$ *generated by the topological closure of the exponential image of the Fourier algebra* $\mathsf{F}(\mathbb{S}^1)$ *is the orientation preserving diffeomorphism group* $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$.

Proof. The Fourier algebra $\mathsf{F}(\mathbb{S}^1)$ is a dense subalgebra of $\mathsf{Vect}(\mathbb{S}^1)$ with respect to the Fréchet topology, i.e. $\overline{\mathsf{F}(\mathbb{S}^1)} = \mathsf{Vect}(\mathbb{S}^1)$. This assertion follows from the fact that any r-times continuously differentiable function can be approximated uniformly by the arithmetical means of the partial sums of its Fourier series (cf. [9], 2.12 Theorem). The exponential mapping is continuous (c.f. Lemma 4.1 in [16], p. 79), hence we have

$$\exp\Big(\mathsf{Vect}(\mathbb{S}^1)\Big) = \exp\big(\overline{\mathsf{F}(\mathbb{S}^1)}\big) \subset \overline{\exp\big(\mathsf{F}(\mathbb{S}^1)\big)} \subset \mathsf{Diff}^\infty_+(\mathbb{S}^1) \tag{10}$$

which gives for the generated groups the relations

$$\Big\langle \exp\big(\mathsf{Vect}(\mathbb{S}^1)\big)\Big\rangle \subset \Big\langle \overline{\exp\big(\mathsf{F}(\mathbb{S}^1)\big)} \Big\rangle \subset \mathsf{Diff}^\infty_+(\mathbb{S}^1). \tag{11}$$

Moreover, the conjugation map $\mathsf{Ad} : \mathsf{Diff}^\infty_+(\mathbb{S}^1) \times \mathsf{Vect}(\mathbb{S}^1)$ satisfies the relation

$$h \exp s\xi\, h^{-1} = \exp s\mathsf{Ad}(h)\xi$$

for every $h \in \mathsf{Diff}^\infty_+(\mathbb{S}^1)$ and $\xi \in \mathsf{Vect}(\mathbb{S}^1)$. Clearly, the Lie algebra $\mathsf{Vect}(\mathbb{S}^1)$ is invariant under conjugation and hence the group $\langle \exp(\mathsf{Vect}(\mathbb{S}^1))\rangle$ is also invariant under conjugation. Therefore $\langle \exp(\mathsf{Vect}(\mathbb{S}^1))\rangle$ is a non-trivial normal subgroup of $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$. On the other hand $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$ is a simple group (cf. [8]) which means that its only non-trivial normal subgroup is itself. Therefore, we have $\langle \exp(\mathsf{Vect}(\mathbb{S}^1))\rangle = \mathsf{Diff}^\infty_+(\mathbb{S}^1)$, and using (11) we get

$$\Big\langle \overline{\exp(\mathsf{F}(\mathbb{S}^1))} \Big\rangle = \mathsf{Diff}^\infty_+(\mathbb{S}^1).$$

5. Holonomy of the standard Funk plane and the Bryant-Shen 2-spheres

Using the results of the preceding chapter we can prove the following statement, which provides a useful tool for the investigation of the closed holonomy group of Finsler 2-manifolds.

Proposition 5.1. *If the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ at a point $x \in M$ of a simply connected Finsler 2-manifold $(M, \mathcal{F})$ contains the Fourier algebra $\mathsf{F}(\mathbb{S}^1)$ on the indicatrix at x, then $\overline{\mathsf{Hol}_x(M)}$ is isomorphic to $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$.*

Proof. Since M is simply connected we have

$$\overline{\mathsf{Hol}_x(M)} \subset \mathsf{Diff}^\infty_+(\mathbb{S}^1). \tag{12}$$

On the other hand, using Proposition 3.1, we get

$$\exp\left(\mathsf{F}(\mathbb{S}^1)\right)\subset\overline{\mathsf{Hol}_x(M)} \;\Rightarrow\; \overline{\exp\left(\mathsf{F}(\mathbb{S}^1)\right)}\subset\overline{\mathsf{Hol}_x(M)} \;\Rightarrow\; \overline{\left\langle \exp\left(\mathsf{F}(\mathbb{S}^1)\right) \right\rangle}\subset\overline{\mathsf{Hol}_x(M)},$$

and from the last relation, using Lemma 4.1, we can obtain that

$$\mathsf{Diff}^\infty_+(\mathbb{S}^1) \subset \overline{\mathsf{Hol}_x(M)}. \tag{13}$$

Comparing (12) and (13) we get the assertion.

Using this proposition we can prove our main result:

Theorem 5.2. *Let $(M, \mathcal{F})$ be a simply connected projectively flat Finsler manifold of constant curvature $\lambda \neq 0$. Assume that there exists a point $x_0 \in M$ such that on $T_{x_0}M$ the induced Minkowski norm is an Euclidean norm, that is $\mathcal{F}(x_0, y) = \|y\|$, and the projective factor at x_0 satisfies $\mathcal{P}(x_0, y) = c\cdot\|y\|$ with $c \in \mathbb{R}$, $c \neq 0$. Then the closed holonomy group $\overline{\mathsf{Hol}_{x_0}(M)}$ at x_0 is isomorphic to $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$.*

Proof. Since $(M, \mathcal{F})$ is a locally projectively flat Finsler manifold of non-zero constant curvature, we can use an (x^1, x^2) local coordinate system centered at $x_0 \in M$, corresponding to the canonical coordinates of the Euclidean space which is projectively related to $(M, \mathcal{F})$. Let (y^1, y^2) be the induced coordinate system in the tangent plane T_xM. In the sequel we identify the tangent plane $T_{x_0}M$ with $\mathbb{R}^2$ by using the coordinate system (y^1, y^2). We will use the Euclidean norm $\|(y^1, y^2)\| = \sqrt{(y^1)^2 + (y^2)^2}$ of $\mathbb{R}^2$ and the corresponding polar coordinate system (e^r, t), too.

Let us consider the curvature vector field ξ at $x_0 = 0$ defined by

$$\xi = R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\Big|_{x=0} = \lambda(\delta^i_2 g_{1m}(0, y)y^m - \delta^i_1 g_{2m}(0, y)y^m)\frac{\partial}{\partial x^i}$$

Since $(M, \mathcal{F})$ is of constant flag curvature, the horizontal Berwald covariant derivative $\nabla_W R$ of the tensor field R vanishes. Therefore the covariant derivative of ξ can be written in the form

$$\nabla_W \xi = R\left(\nabla_k\left(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}\right)\right) W^k.$$

Since

$$\nabla_k\left(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}\right) = \left(G^1_{k1} + G^2_{k2}\right)\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$$

we obtain $\nabla_W \xi = \left(G^1_{k1} + G^2_{k2}\right) W^k \xi$. Using (4) we can express $G^m_{km} = 3\frac{\partial P}{\partial y^k} = 3c\frac{y^k}{\|y\|}$ and hence

$$\nabla_k \xi = 3\frac{\partial P}{\partial y^k}\xi = 3c\frac{y^k}{\|y\|}\xi,$$

where we use the notation $\nabla_k = \nabla_{\frac{\partial}{\partial x^k}}$. Moreover we have

$$\nabla_j \left(\frac{\partial \mathcal{P}}{\partial y^k}\right) = \frac{\partial^2 \mathcal{P}}{\partial x^j \partial y^k} - G^m_j \frac{\partial^2 \mathcal{P}}{\partial y^m \partial y^k} = \frac{\partial^2 \mathcal{P}}{\partial x^j \partial y^k} - \mathcal{P}\frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^j},$$

and hence

$$\nabla_j \left(\nabla_k \xi\right) = 3\left\{\frac{\partial^2 \mathcal{P}}{\partial x^j \partial y^k} - \mathcal{P}\frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^j} + 3\frac{\partial \mathcal{P}}{\partial y^k}\frac{\partial \mathcal{P}}{\partial y^j}\right\}\xi.$$

According to Lemma 8.2.1, equation (8.25) in [4], p. 155, we obtain

$$\frac{\partial^2 \mathcal{P}}{\partial x^j \partial y^k} = \frac{\partial \mathcal{P}}{\partial y^j}\frac{\partial \mathcal{P}}{\partial y^k} + \mathcal{P}\frac{\partial^2 \mathcal{P}}{\partial y^j \partial y^k} - \frac{\lambda}{2}\frac{\partial^2 \mathcal{F}^2}{\partial y^j \partial y^k}.$$

Using the assumptions on $\mathcal{F}$ and on the projective factor $\mathcal{P}$ we can get at x_0

$$\nabla_j \left(\nabla_k \xi\right) = 3\left(4c^2\frac{\partial \mathcal{F}}{\partial y^j}\frac{\partial \mathcal{F}}{\partial y^k} - \frac{\lambda}{2}\frac{\partial^2 \mathcal{F}^2}{\partial y^j \partial y^k}\right)\xi$$

and hence

$$\nabla_j \left(\nabla_k \xi\right) = 3\left(4\,c^2\frac{y^j y^k}{\|y\|^2} - \lambda\,\delta^{jk}\right)\xi,$$

where $\delta^{jk} \in \{0,1\}$ such that $\delta^{jk} = 1$ if and only if $j = k$.
Let us introduce polar coordinates $y^1 = r\cos t$, $y^2 = r\sin t$ in the tangent space $T_{x_0}M$. We can express the curvature vector field, its first and second covariant derivatives along the indicatrix curve $\{(\cos t, \sin t);\ 0 \le t < 2\pi\}$ as follows:

$$\xi = \lambda\frac{d}{dt}, \quad \nabla_1\xi = 3c\lambda\cos t\frac{d}{dt}, \quad \nabla_2\xi = -3c\lambda\sin t\frac{d}{dt}, \quad \nabla_1(\nabla_2\xi) = 12\,c^2\lambda\sin 2t\frac{d}{dt},$$

$$\nabla_1(\nabla_1\xi) = \lambda\left(12\,c^2\cos^2 t - \lambda\right)\frac{d}{dt}, \quad \nabla_2(\nabla_2\xi) = \lambda\left(12\,c^2\sin^2 t - \lambda\right)\frac{d}{dt}.$$

Since $c\,\lambda \neq 0$, the vector fields

$$\frac{d}{dt}, \quad \cos t\,\frac{d}{dt}, \quad \sin t\,\frac{d}{dt}, \quad \cos t\sin t\,\frac{d}{dt}, \quad \cos^2 t\,\frac{d}{dt}, \quad \sin^2 t\,\frac{d}{dt}$$

are contained in the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_0}(M)$. It follows that the generator system

$$\left\{\frac{d}{dt}, \quad \cos t\,\frac{d}{dt}, \quad \sin t\,\frac{d}{dt}, \quad \cos 2t\,\frac{d}{dt}, \quad \sin 2t\,\frac{d}{dt}\right\}$$

of the Fourier algebra $\mathsf{F}(\mathbb{S}^1)$ (c.f. equation (9)) is contained in the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_0}(M)$. Hence the assertion follows from Proposition 5.1.

We remark, that the standard Funk plane and the Bryant-Shen 2-spheres are connected, projectively flat Finsler manifolds of nonzero constant curvature. Moreover, in each of them, there exists a point $x_0 \in M$ and an adapted local coordinate system centered at x_0 with the following properties: the Finsler norm $\mathcal{F}(x_0, y)$ and the projective factor $\mathcal{P}(x_0, y)$ at x_0 are given by $\mathcal{F}(x_0, y) = ||y||$ and by $\mathcal{P}(x_0, y) = c \cdot ||y||$ with some constant $c \in \mathbb{R}$, $c \neq 0$, where $||y||$ is an Euclidean norm in the tangent space at x_0. Using Theorem 5.2 we can obtain

Theorem 5.3. *The closed holonomy groups of the standard Funk plane and of the Bryant-Shen 2-spheres are maximal, that is diffeomorphic to the orientation preserving diffeomorphism group of* $\mathbb{S}^1$.

Open problems

In contrast to the Riemannian case, there are only few results about the holonomy of Finsler spaces. Even for the most commonly investigated Finsler metrics (like (α,β) metrics, m-root metrics etc.) the holonomy group is not known. It would be very useful to have explicitely computed examples. The curvature algebra could be used to find estimates on the dimension of the holonomy group [12].

The new phenomenon with respect to the Riemann holonomy is that the Finsler holonomy can be infinite dimensional [15, 12]. There are also examples however, where the Finsler holonomy group is finite dimensional (e.g. Berwald manifolds). It would be extremely interesting to characterise Finsler structures with finite, respectively infinite dimensional holonomy group. Even partial results would be interesting.

In Finsler geometry, any information about the geometric properties of Landsberg manifolds are very important. It is well known that the holonomy group of Landsberg manifolds is contained in the isometry group of the indicatrix and hence it is a compact group [10]. Can we get more information about these holonomy groups? Is there a Landsberg manifold whose holonomy group is not the holonomy group of any Riemannian manifold?

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