BIOMICS

Classification of Simple Real Lie Algebras

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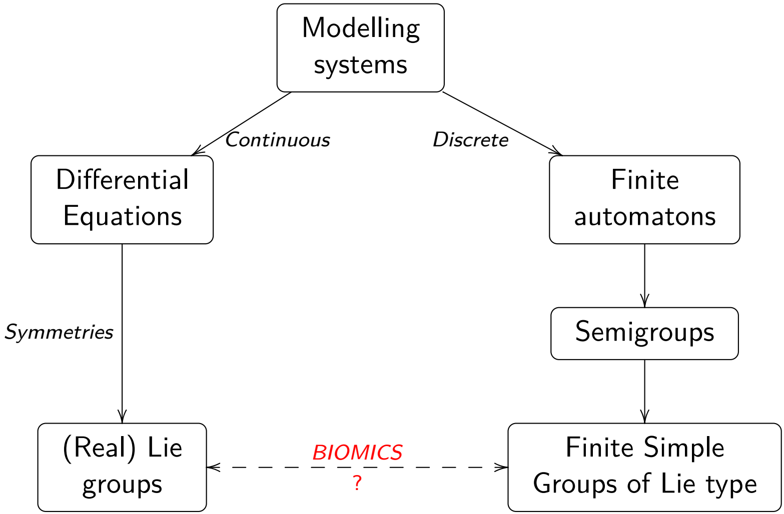
Second BIOMICS Summer Workshop
18-20 June 2014, University of St Andrews, Scotland

SEVENTH FRAMEWORK PROGRAMME


Modelling systems
Continuous
Discrete
Differential Equations
Finite automatons
Symmetries
Semigroups
(Real) Lie groups
BIOMICS
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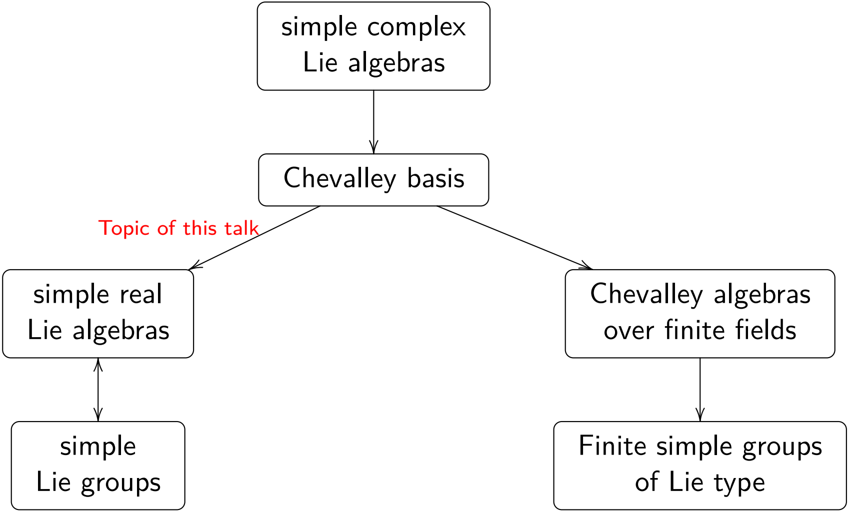
Finite Simple Groups of Lie type

Connections between Lie algebras and Lie groups over various fields



Structure of Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ or over $\mathbb{C}$.

- $\mathfrak{g}$ is semisimple if it does not contain any non-trivial solvable ideal, $\mathfrak{g}$ is simple if it does not contain any non-trivial ideal.
- Lévi-Malcev theorem: $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ where $\mathfrak{r}$ is solvable, while $\mathfrak{s}$ is semisimple.
- Any semisimple Lie algebra can be uniquely decomposed into a direct product of simple Lie algebras.
- The simple components of this decomposition are orthogonal to each other with respect to the Killing form.

Solvable Lie algebras have simple structure, but it is impossible to classify all of them. On the other hand, simple Lie algebras are difficult, but we know all of them!

Complexification and real forms

- Let $\mathfrak{g}_0$ be a real Lie algebra. Its complexification $\mathfrak{g} = \mathfrak{g}_0(\mathbb{C}) := \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ as a vector space with extended Lie bracket

$$[x_1 + iy_1, x_2 + iy_2] := [x_1, y_1] - [x_2, y_2] + i([x_1, y_2] + [x_2, y_1])$$

- If $\mathfrak{g}$ is a complex Lie algebra, then its realification $\mathfrak{g}_{\mathbb{R}}$ is just $\mathfrak{g}$ viewed as a real Lie algebra.
- If $\{v_1, \ldots, v_n\}$ is a basis of $\mathfrak{g}_0$, then it is also a basis of the complex Lie algebra $\mathfrak{g} = \mathfrak{g}_0(\mathbb{C})$ and $\{v_1, \ldots, v_n, iv_1, \ldots, iv_n\}$ is a basis of the real Lie algebra $\mathfrak{g}_{\mathbb{R}}$. So

$$\dim_{\mathbb{R}} \mathfrak{g}_0 = \dim_{\mathbb{C}} \mathfrak{g}, \quad \dim_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}} = 2 \dim_{\mathbb{C}} \mathfrak{g}.$$

- If $\mathfrak{g}$ is any complex Lie algebra, then a real form of $\mathfrak{g}$ is a real subalgebra $\mathfrak{g}_0 \leq \mathfrak{g}_{\mathbb{R}}$ s.t. $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. Then $\mathfrak{g} \simeq \mathfrak{g}_0(\mathbb{C})$.

The Killing form

- If $\mathfrak{g}$ is a Lie algebra and $X \in \mathfrak{g}$, then the map $\operatorname{ad} X : \mathfrak{g} \mapsto \mathfrak{g}$ is defined as
$$\operatorname{ad} X(Y) := [X, Y]$$
Then $\operatorname{ad} : \mathfrak{g} \mapsto \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra homomorphism.
- The Killing form on a (real or complex) Lie algebra $\mathfrak{g}$ is a symmetric bilinear function on $\mathfrak{g}$ and defined as
$$k(X, Y) = \operatorname{Tr}(\operatorname{ad} X \cdot \operatorname{ad} Y) \qquad X, Y \in \mathfrak{g}$$
- $\mathfrak{g}$ is semisimple $\iff$ its Killing form k is non-degenerate.
- If $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$ then $k_{\mathfrak{g}}|\mathfrak{g}_0 = k_{\mathfrak{g}_0}$.
- $\mathfrak{g}_0$ is semisimple $\iff$ $\mathfrak{g} = \mathfrak{g}_0(\mathbb{C})$ is semisimple.

The simple real Lie algebras

- If $\mathfrak{g}$ is a simple complex algebra then
  - $\mathfrak{g}_{\mathbb{R}}$ is a simple real Lie algebra with $\mathfrak{g}_{\mathbb{R}}(\mathbb{C}) \simeq \mathfrak{g} \oplus \mathfrak{g}$.
  - Any real form $\mathfrak{g}_0$ of $\mathfrak{g}$ is a simple real Lie algebra.
- Conversely, if $\mathfrak{g}_0$ is a simple real Lie algebra then one of the following holds.
  - There is a complex structure on $\mathfrak{g}_0$ i.e. $\mathfrak{g}_0 = \mathfrak{g}_{\mathbb{R}}$ for some complex simple Lie algebra $\mathfrak{g}$.
  - $\mathfrak{g}_0(\mathbb{C})$ is a simple complex Lie algebra (with real form $\mathfrak{g}_0$).

Thus, Classifying the simple real Lie algebras $\Longleftrightarrow$ Finding all non-isomorphic real forms of the simple complex Lie algebras

The structure of a complex simple Lie algebra $\mathfrak{g}$

Let $\mathfrak{g}$ be a simple (or semisimple) Lie algebra over $\mathbb{C}$. Then the following holds

- There is a maximal toral subalgebra (Cartan subalgebra) $\mathfrak{h} \leq \mathfrak{g}$ such that ad h, $h \in \mathfrak{h}$ are simultaneously diagonalizable.
- The root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

  where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, $\Phi \subseteq \mathfrak{h}^*$ is a system of roots and $\mathfrak{g}_\alpha$ is the root space corresponding to $\alpha \in \Phi$.
- $\dim g_\alpha = 1$ and $[h, x] = \alpha(h) \cdot x \ \forall \alpha \in \Phi, \ h \in \mathfrak{t}, \ x \in \mathfrak{g}_\alpha$.
- If $\alpha \in \Phi$ then $-\alpha \in \Phi$ (there are some other nice properties)

- There is a $\Pi \subseteq \Phi$ is a fundamental system of roots.
- $\Phi = \Phi^+ \cup \Phi^-$ is the decomposition of Φ to the system of positive and negative roots with respect to Π.
- There is a Chevalley basis of the complex Lie algebra $\mathfrak{g}$:

$$CB = \{h_\alpha,\ \alpha \in \Pi,\ e_\alpha,\ \alpha \in \Phi\}$$

  where $\{h_\alpha,\ \alpha \in \Pi\}$ is a basis of $\mathfrak{h}$ and $\mathfrak{g}_\alpha = \mathbb{C}e_\alpha$, such that all the structure constants of the Lie algebra with respect to this basis are reals (even integers).
- The canonical generator set for $\mathfrak{g}$

$$\{h_\alpha,\ \alpha \in \Pi,\ e_\alpha,\ e_{-\alpha}\ \alpha \in \Pi\} \subseteq CB$$

- The Killing form is non-degenerate on $\mathfrak{h}$. With its help one can define a non-degenerate symmetric bilinear form $(. , .)$ on $\mathfrak{h}^*$.
- $(. , .)$ is positive definite on the real subspace $E := \langle \Phi \rangle$ of $\mathfrak{h}^*$, E is a real Euclidean space with scalar product $(. , .)$.
- For every $\alpha, \beta \in \Pi$ we have the Cartan integers

$$\langle \alpha, \beta \rangle := \frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}$$

- and the Dynkin diagrams
  - vertices: Π
  - edges : For $\alpha, \beta \in \Pi$ the number of edges between α and β is $\langle \alpha, \beta \rangle \cdot \langle \beta, \alpha \rangle$ directed to the shorter root.

The Dynkin diagrams of simple Lie algebras

Table: Classical Lie algebras

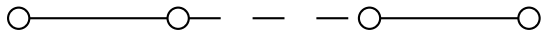
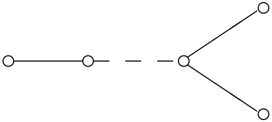
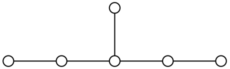
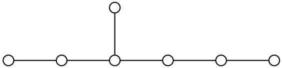
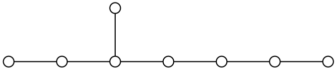
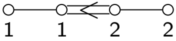
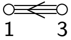
Type	Diagram	over $\mathbb{C}$
A_l		$\mathfrak{sl}_{l+1}(\mathbb{C})$
B_l		$\mathfrak{so}_{2l+1}(\mathbb{C})$
C_l		$\mathfrak{sp}_l(\mathbb{C})$
D_l		$\mathfrak{so}_{2l}(\mathbb{C})$

Table: Exceptional Lie algebras

Type	Diagram
E_6	
E_7	
E_8	
F_4	
G_2	

Existence and Isomorphism theorems

Existence Theorem

For every Dynkin diagram there is a simple complex Lie algebra with this Dynkin diagram.

Uniqueness Theorem

Let $(\mathfrak{g}, \mathfrak{h}, \Phi, \Pi)$ and $(\mathfrak{g}', \mathfrak{h}', \Pi' \subseteq \Phi')$ be two simple complex Lie algebras with Cartan subalgebras and associated root systems and fundamental systems.
If $\varphi : \Pi \mapsto \Pi'$ is a bijection respecting the Cartan integers (i.e. it is an isomorphism of the Dynkin diagrams) then φ induces an isomorphism $\pi : \mathfrak{h} \mapsto \mathfrak{h}'$ (via the Killing form).
Moreover, choosing non-zero $x_\alpha \in \mathfrak{g}_\alpha,\ x'_\alpha \in \mathfrak{g}'_\alpha$ for each $\alpha \in \Pi$, π extends uniquely to an isomorphism $\pi : \mathfrak{g} \mapsto \mathfrak{g}'$ such that $\pi(x_\alpha) = x'_\alpha$ for each x_α.

Split real forms

Let $\mathfrak{g}$ be a simple complex Lie algebra with real form $\mathfrak{g}_0$.

- The conjugation $\sigma : \mathfrak{g} \mapsto \mathfrak{g}$ with respect to $\mathfrak{g}_0$ is

$$\sigma(x+iy) = x - iy \qquad \forall\, x, y \in \mathfrak{g}_0.$$

  Then
$$\mathfrak{g}_0 = \mathfrak{g}^\sigma = \{x \in \mathfrak{g} \,|\, \sigma(x) = x\}.$$

- A split real form $\mathfrak{g}_0$ of $\mathfrak{g}$ can be defined by choosing a Chevalley basis $CB = \{h_\alpha,\ \alpha \in \Pi,\ e_\alpha,\ \alpha \in \Phi\}$ of $\mathfrak{g}$ and taking the real subspace generated by it. Thus,

$$\mathfrak{g}_0 = \sum_{\alpha\in\Pi} \mathbb{R}h_\alpha + \sum_{\alpha\in\Phi} \mathbb{R}e_\alpha$$

Compact real form

- $\mathfrak{u}_0$ is a compact real form of $\mathfrak{g}$ if its Killing form is negative definite. ($\Longleftrightarrow$ it is the Lie algebra of a compact Lie group.)
- For the canonical generator set $\{h_\alpha, e_\alpha, e_{-\alpha} \,|\, \alpha \in \Pi\}$ the map

$$\omega(h_\alpha) = -h_\alpha, \ \omega(e_\alpha) = -e_{-\alpha}, \ \omega(e_{-\alpha}) = -e_\alpha$$

  on the canonical generators extends uniquely to an automorphism $\omega \in \operatorname{Aut}\mathfrak{g}$ with $\omega^2 = \mathrm{id}$. ω is called a Weyl involution.
- Let $\mathfrak{g}_0$ be the split real form of $\mathfrak{g}$ and σ the conjugation with respect to $\mathfrak{g}_0$. Let $\tau = \omega\sigma$. By choosing

$$\mathfrak{u}_0 := \mathfrak{g}^\tau = \{x \in \mathfrak{g} \,|\, \tau(x) = x\}$$

  we get a compact real form of $\mathfrak{g}$.
- By using the same CB as in the split real form

$$\mathfrak{u}_0 = \sum_{\alpha\in\Pi} \mathbb{R} i h_\alpha + \sum_{\alpha\in\Phi^+} \mathbb{R}(e_\alpha - e_{-\alpha}) + \sum_{\alpha\in\Phi^+} \mathbb{R} i(e_\alpha + e_{-\alpha})$$

Cartan involutions

Let $\mathfrak{g}$ be a complex (semi)simple algebra with compact real form $\mathfrak{u}_0$ and arbitrary real form $\mathfrak{g}_0$.

- An involution of $\mathfrak{g}_0$ is a map $\theta \in \mathrm{Aut}(\mathfrak{g}_0)$ with $\theta^2 = \mathrm{id}$.
- A Cartan involution of $\mathfrak{g}_0$ is an involution θ of $\mathfrak{g}_0$ such that

$$k_\theta(u, v) := -k(u, \theta(v))$$

  is positive definite.
- $\mathfrak{g}_0$ is a compact real form of $\mathfrak{g}$ $\iff$ the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ is a Cartan involution of $\mathfrak{g}_\mathbb{R}$.
- Every real semisimple Lie algebra $\mathfrak{g}_0$ has a Cartan involution.
- Any two Cartan involutions of $\mathfrak{g}_0$ are conjugate via $\mathrm{Int}(\mathfrak{g}_0)$. Thus, any two compact real forms of $\mathfrak{g}$ are isomorphic.

Cartan decomposition

- A Cartan decomposition of $\mathfrak{g}_0$ is a vector space direct sum $\mathfrak{g}_0 = \mathfrak{n}_0 \oplus \mathfrak{p}_0$ with properties
  - $$[\mathfrak{n}_0, \mathfrak{n}_0] \subseteq \mathfrak{n}_0, \quad [\mathfrak{n}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0, \quad [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{n}_0.$$
  - the Killing form of $\mathfrak{g}_0$
$$k_{\mathfrak{g}_0} \text{ is } \begin{cases} \text{negative definite on } \mathfrak{n}_0 \\ \text{positive definite on } \mathfrak{p}_0 \end{cases}$$
- Correspondence between Cartan involutions and Cartan decompositions
$$\theta \Longrightarrow \begin{cases} \mathfrak{n}_0 & = \{x \in \mathfrak{g}_0 \mid \theta(x) = x\} \\ \mathfrak{p}_0 & = \{x \in \mathfrak{g}_0 \mid \theta(x) = -x\} \end{cases}$$
$$\theta = \begin{cases} +\mathrm{id} & \text{on } \mathfrak{n}_0 \\ -\mathrm{id} & \text{on } \mathfrak{p}_0 \end{cases} \Longleftarrow \mathfrak{g} = \mathfrak{n}_0 \oplus \mathfrak{p}_0$$
- If $\mathfrak{g}_0 = \mathfrak{n}_0 \oplus \mathfrak{p}_0$ is a Cartan decomposition then $\mathfrak{n}_0$ and $\mathfrak{p}_0$ are orthogonal with respect to $k_{\mathfrak{g}_0}$ and k_θ.

Cartan subalgebras

Let $\mathfrak{g}_0$ be a real semisimple algebra with Cartan involution θ and corresponding Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$.

- $\mathfrak{h}_0 \subseteq \mathfrak{g}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$ if $\mathfrak{h}_0(\mathbb{C})$ is a Cartan subalgebra of $\mathfrak{g} = \mathfrak{g}_0(\mathbb{C})$.
- Any Cartan subalgebra of $\mathfrak{g}_0$ is conjugate via $\mathrm{Int}(\mathfrak{g}_0)$ to a θ-stable Cartan subalgebra of $\mathfrak{g}_0$ (i.e. with $\mathfrak{h}_0$ satisfying $\theta(\mathfrak{h}_0) = \mathfrak{h}_0$).

From now on, let $\mathfrak{h}_0 \subseteq \mathfrak{g}_0$ be a θ-stable Cartan subalgebra.

- We have the decomposition

$$\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \text{ with } \mathfrak{t}_0 = \mathfrak{h}_0 \cap \mathfrak{k}_0 \text{ and } \mathfrak{a}_0 = \mathfrak{h}_0 \cap \mathfrak{p}_0.$$

- $\dim \mathfrak{t}_0$ is called the compact dimension of $\mathfrak{h}_0$
  $\dim \mathfrak{a}_0$ is called the noncompact dimension of $\mathfrak{h}_0$

- The (θ-stable) $\mathfrak{h}_0$ is called maximally compact if its compact dimension is maximal.
- If $\mathfrak{t}_0 \subseteq \mathfrak{n}_0$ is a maximal abelian subspace then $\mathfrak{h}_0 := Z_{\mathfrak{g}_0}(\mathfrak{t}_0)$ is a θ-stable maximally compact Cartan subalgebra of $\mathfrak{g}_0$.
- All the maximally compact θ-stable Cartan subalgebras of $\mathfrak{g}_0$ are conjugate via $\mathrm{Aut}(\mathfrak{g}_0)$.

From now on, let $\mathfrak{h}_0$ be maximally compact.

- Let α be a root of $\mathfrak{g} = \mathfrak{g}_0(\mathbb{C})$ with respect to the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0 \oplus i\mathfrak{h}_0$. Then α is real valued on $\mathfrak{a}_0 \oplus i\mathfrak{t}_0$.
  - α is real if it is real valued on $\mathfrak{h}_0$ (i.e. it vanishes on $\mathfrak{t}_0$)
  - α is imaginary if it is purely imaginary valued on $\mathfrak{h}_0$ (i.e. it vanishes on $\mathfrak{a}_0$)
  - α is complex otherwise.
- $\mathfrak{h}_0$ is maximally compact $\iff$ there are no real roots.

Vogan diagrams

Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ be a root space decomposition of $\mathfrak{g} = \mathfrak{g}_0(\mathbb{C})$ with Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0(\mathbb{C})$.

- If $\alpha \in \mathfrak{h}^*$ is a root, then $\theta\alpha(h) := \alpha(\theta^{-1}h)$ is also a root.
- Let $\{x_1, \ldots, x_n\}$ and $\{x_{n+1}, \ldots, x_l\}$ be basises of $i\mathfrak{t}_0$ and $\mathfrak{a}_0$, respectively. Then $\{x_1, \ldots, x_l\}$ is a basis of $i\mathfrak{t}_0 \oplus \mathfrak{a}_0$. For $\alpha, \beta \in \Phi$ let

$$\alpha > \beta \iff \exists t : \alpha(x_t) > \beta(x_t) \text{ and } \alpha(x_s) = \beta(x_s)\ \forall s < t$$

  Then $>$ is an ordering on $\Phi \implies \Phi^+, \Phi^-, \Pi \subseteq \Phi$
- θ is identical on $\mathfrak{t}_0$ and there are no real roots $\implies$

$$\theta(\Phi^+) = \Phi^+ \text{ and } \theta(\Pi) = \Pi.$$

- θ fixes the imaginary roots while it permutes the complex roots in 2-cycles.

- If $\alpha \in \Phi$ is imaginary then $\mathfrak{g}_\alpha$ is θ-stable
  $\Rightarrow \mathfrak{g}_\alpha = (\mathfrak{g}_\alpha \cap \mathfrak{n}_0) \oplus (\mathfrak{g}_\alpha \cap \mathfrak{p}_0) \Rightarrow \mathfrak{g}_\alpha \subseteq \mathfrak{n}_0$ or $\mathfrak{g}_\alpha \subseteq \mathfrak{p}_0$.
  - α is called compact if $\mathfrak{g}_\alpha \subseteq \mathfrak{n}_0$
  - α is called noncompact if $\mathfrak{g}_\alpha \subseteq \mathfrak{p}_0$
- The Vogan diagram corresponding to $(\mathfrak{g}_0, \mathfrak{h}_0, \Phi^+)$ is the following
  - We start with the Dynkin diagram of $\mathfrak{g}_0(\mathbb{C})$
  - The vertices of the 2-element orbits of θ are connected by an arrow $\leftrightarrow$
  - The noncompact imaginary roots are painted.
- The "empty" Dynkin diagrams correspond to compact real forms.

Existence and uniqueness theorems

Existence theorem

For any abstract Vogan diagram, there is a real semisimple algebra $\mathfrak{g}_0$, a Cartan involution θ, a maximally compact θ-stable Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ and a positive system Φ^+ such that the triple $(\mathfrak{g}_0, \mathfrak{h}_0, \Phi^+)$ defines the given Vogan diagram.

Uniqueness theorem

If $(\mathfrak{g}_0, \mathfrak{h}_0, \Phi^+)$ and $(\mathfrak{g}_0', \mathfrak{h}_0', \Phi'^+)$ are two isomorphic triples, then they have the same Vogan diagram.

Conversely, if $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ are real semisimple algebras having triples $(\mathfrak{g}_0, \mathfrak{h}_0, \Phi^+)$ and $(\mathfrak{g}_0', \mathfrak{h}_0', \Phi'^+)$ with the same Vogan diagram then $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ are isomorphic.

Unfortunately, different Vogan diagrams can define isomorphic Lie algebras. This redundancy can be resolved by the following Theorem

Borel and de Siebenthal Theorem

If $\mathfrak{g}_0$ is a noncomplex simple real Lie algebra with triple $(\mathfrak{g}_0, \mathfrak{h}_0, \Phi^+)$ then one can choose a fundamental system $\Pi' \subseteq \Phi$ defining positive system Φ' such that there is at most one painted root in the Vogan diagram defined by $(\mathfrak{g}_0, \mathfrak{h}_0, \Phi'^+)$.

If the action of θ on the Vogan diagram is the identity, let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ and let $\{\omega_1, \ldots, \omega_l\}$ be the dual basis i.e. which satisfies $(\omega_i, \alpha_j) = \delta_{ij}$. Then the single painted root α_i may be chosen s.t. there is no j with $(\omega_i - \omega_j, \omega_j) > 0$.

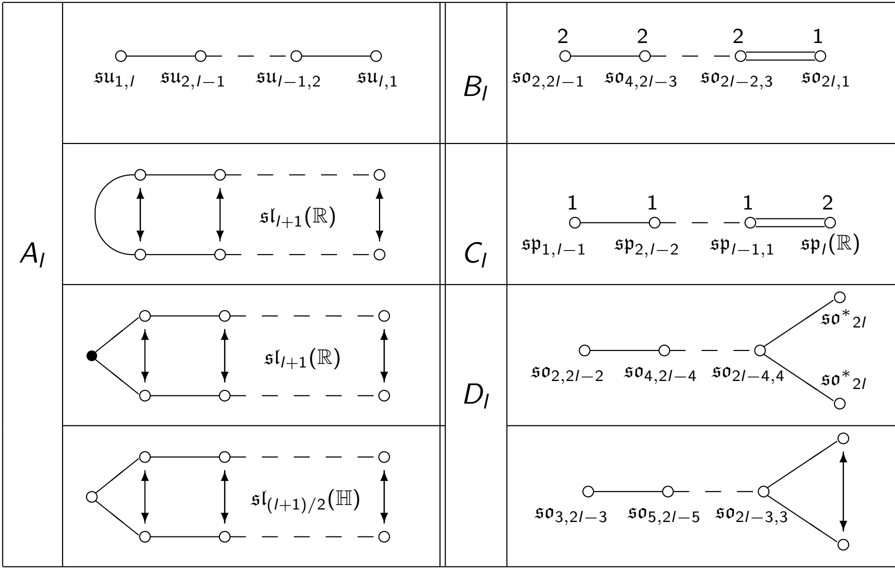


Table: Vogan diagrams of noncompact classical Lie algebras

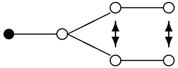
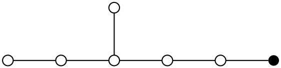
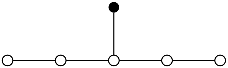
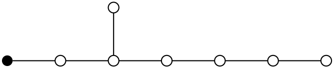
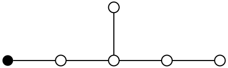
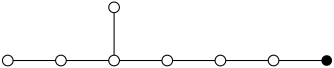
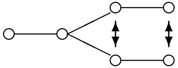
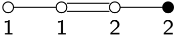
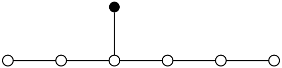
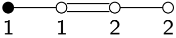
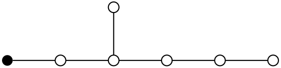
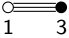
E I		EVII	
E II		EVIII	
E III		EIX	
E IV		FI	
E V		FII	
E VI		G	

Table: Vogan diagrams of noncompact exceptional Lie algebras

The classical Lie algebras

Let I_n be the identity matrix and

$$J_{n,n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}, \quad K_{n,n} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

Classical Lie algebras over $\mathbb{C}$

$$\mathfrak{sl}_n(\mathbb{C}) = \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid \operatorname{Tr} X = 0\}$$
$$\mathfrak{so}_n(\mathbb{C}) = \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid X + X^T = 0\}$$
$$\mathfrak{sp}_n(\mathbb{C}) = \{X \in \mathfrak{gl}_{2n}(\mathbb{C}) \mid X^T J_{n,n} + J_{n,n} X = 0\}$$

Classical Lie algebras over $\mathbb{R}$

- Type A_l :

$$\mathfrak{sl}_n(\mathbb{R}) = \{X \in \mathfrak{gl}_n(\mathbb{R}) \mid \operatorname{Tr} X = 0\}$$ split real form

$$\mathfrak{sl}_n(\mathbb{H}) = \{X \in \mathfrak{gl}_n(\mathbb{H}) \mid \operatorname{Re} \operatorname{Tr} X = 0\}$$

$$\mathfrak{su}_{p,q} = \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid X^* I_{p,q} + I_{p,q} X = 0\}$$ compact real form for $p = 0$

- Type B_l and D_l:

$$\mathfrak{so}_{p,q} = \{X \in \mathfrak{gl}_{p+q}(\mathbb{R}) \mid X^* I_{p,q} + I_{p,q} X = 0\}$$
$$\mathfrak{so}^*_{2l} = \{X \in \mathfrak{su}_{n,n} \mid X^T K_{n,n} + K_{n,n} X = 0\}$$

- Type C_l:

$$\mathfrak{sp}_{p,q} = \{X \in \mathfrak{gl}_{p+q}(\mathbb{H}) \mid X^* I_{p,q} + I_{p,q} X = 0\}$$
$$\mathfrak{sp}_n(\mathbb{R}) = \{X \in \mathfrak{gl}_{2n}(\mathbb{R}) \mid X^T J_{n,n} + J_{n,n} X = 0\}$$

Let $\mathfrak{g} = \mathfrak{sl}_{l+1}(\mathbb{C})$ and $\{E_{ij} \,|\, 1 \leq i, j \leq l+1\}$ be the canonical basis of $\mathfrak{gl}_{l+1}(\mathbb{C})$.

- Cartan subalgebra $\mathfrak{h} = \{\mathrm{diag}(x_1, \ldots, x_{l+1}) \,|\, \sum x_i = 0\}$

Let $e_i \in \mathfrak{h}^*,\ e_i : \mathrm{diag}(x_1, \ldots, x_{l+1}) \mapsto x_i$.

- Roots $\alpha_{ij} = e_i - e_j \in \mathfrak{h}^*,\ \alpha_{ij}(\mathrm{diag}(x_1, \ldots, x_{l+1})) = x_i - x_j$
- Positive, negative and fundamental roots

$$\Phi^+ = \{\alpha_{ij} \,|\, i < j\}, \quad \Phi^- = \{\alpha_{ij} \,|\, i > j\}, \quad \Pi = \{\alpha_{i,i+1} \,|\, 1 \leq i \leq l\}.$$

- Chevalley basis

$$\{H_i = E_{ii} - E_{i+1,i+1} \,|\, 1 \leq i \leq l\} \cup \{E_{ij} \,|\, i \neq j\}$$

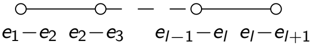
- The scalar product on $\langle \Phi \rangle_{\mathbb{R}}$ is defined by

$$(\alpha_{i,i+1}, \alpha_{j,j+1}) = \alpha_{i,i+1}(H_j) = \begin{cases} 2 & \text{for } k = l \\ -1 & \text{for } |k-l| = 1 \\ 0 & \text{else} \end{cases}$$

- The Cartan matrix A_{kl} containing the Cartan integers $\langle \alpha_{ij}, \alpha_{kl} \rangle = \frac{2(\alpha_{ij},\alpha_{kl})}{(\alpha_{kl},\alpha_{kl})}$

$$A_l = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

- Hence we get the Dynkin diagram



- The split real form of $\mathfrak{g} = \mathfrak{sl}_{l+1}(\mathbb{C})$ is

$$\mathfrak{sl}_{l+1}(\mathbb{R}) = \langle H_i,\ 1 \leq i \leq l, E_{ij},\ i \neq j\rangle_{\mathbb{R}}$$

  The corresponding conjugation is $\sigma : X \mapsto \overline{X}$.
- The Weyl involution can be given as $\omega = X \mapsto -X^T$. Hence the Cartan involution $\theta = \omega\sigma : X \mapsto -\overline{X}^T = -X^*$ defines the compact real form

$$\mathfrak{g}^\theta = \{X \in \mathfrak{sl}_{l+1}(\mathbb{C}) \,|\, X = -X^*\} = \mathfrak{su}_{l+1}(\mathbb{C})$$

  The corresponding Vogan diagram is just the "empty" Dynkin diagram.

- For $p+q=l+1, 1 \le p, q \le l$ let

$$\mathfrak{su}_{p,q} = \{X \in \mathfrak{sl}_{l+1}(\mathbb{C}) \,|\, X^* I_{p,q} + I_{p,q} X = 0\} =$$
$$= \left\{ X = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \,\middle|\, A = -A^*,\ D = -D^* \right\}$$

  - The complexification of $\mathfrak{su}_{p,q}$ is $\mathfrak{sl}_{l+1}(\mathbb{C})$, so it is a real form of $\mathfrak{sl}_{l+1}(\mathbb{C})$.
  - The Cartan decomposition of $\mathfrak{su}_{p,q}$ corresponding to the Cartan involution $\theta(X) = -X^*$ is

$$\mathfrak{n}_0 = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \qquad \mathfrak{p}_0 = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$$

  - The diagonal matrices in $\mathfrak{su}_{p,q}$ are all in $\mathfrak{n}_0$, so it is a maximally compact Cartan subalgebra.
  - There are no complex roots $\Rightarrow \theta$ trivially acts on Π.
  - The only simple root whose root space is not in $\mathfrak{n}_0$ is $e_p - e_{p+1} \Rightarrow$ the only painted root is the p-th.
  - We get he Vogan diagram of type A_l with trivial automorphism and with only the p-th vertex painted.