Holonomy theory of Finsler manifolds

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1 Introduction

The notion of the holonomy group of a Riemannian or Finslerian manifold can be introduced in a very natural way: it is the group generated by parallel translations along loops. In contrast to the Finslerian case, the Riemannian holonomy groups have been extensively studied. One of the earliest fundamental results is the theorem of Borel and Lichnerowicz [3] from 1952, claiming that the holonomy group of a simply connected Riemannian manifold is a closed Lie subgroup of the orthogonal group $O(n)$. By now, the complete classification of Riemannian holonomy groups is known. The holonomy group of a Finsler manifold is the subgroup of the diffeomorphism group of an indicatrix, generated by canonical homogeneous (nonlinear) parallel translations along closed loops. Before our investigation (c.f. [24], [25], [26], [27], [28], [29]), the holonomy groups of non-Riemannian Finsler manifolds have been described only in special cases: for Berwald manifolds there exist Riemannian metrics with the same holonomy group (cf. Z. I. Szabó, [39]), for positive definite Landsberg manifolds the holonomy groups are compact Lie groups consisting of isometries of the indicatrix with respect to an induced Riemannian metric (cf. L. Kozma, [19], [20]). A thorough study of holonomy groups of homogeneous (nonlinear) connections was initiated by W. Barthel in his basic work [2] in 1963; he gave a construction for a holonomy algebra of vector fields on the tangent space. A general setting for the study of infinite dimensional holonomy groups and holonomy algebras of nonlinear connections was initiated by P. Michor in [23]. However the introduced holonomy algebras could not be used to estimate the dimension of the holonomy group since their tangential properties to the holonomy group were not clarified.

In this paper we construct and investigate tangent Lie algebras to the holonomy group and to show that the dimension of these tangent algebras in many cases is greater then the possible dimensions of Riemannian holonomy groups.

In the second section we collect the necessary definitions and constructions of spray and Finsler geometry. The third section is devoted to the investigation of tangential properties of subalgebras of the Lie algebra of vector fields on a manifold to the infinite dimensional diffeomorphis group of this manifold. Particularly we consider the case if the manifold is compact, in this case the diffeomorphism group is an infinite dimensional Lie group modeled on the Lie algebra of vector fields on the manifold.

In Section 4 we introduce the notion of curvature algebra of a Finsler manifold consisting of of tangent vector fields on the indicatrix, which is a generalization of the matrix group generated by curvature operators of a Riemannian manifold. We show that the vector fields belonging to the curvature algebra are tangent to one-parameter families of diffeomorphisms contained in the holonomy group. We prove that for a positive definite non-Riemannian Finsler manifold of non-zero constant curvature with dimension $n > 2$ the dimension of the curvature algebra is strictly greater than the dimension of the orthogonal group acting on the tangent space and hence it can not be a compact Lie group. In addition, we provide an example of a left invariant singular (non y-global) Finsler metric of Berwald-Moór-type on the 3-dimensional Heisenberg group which has infinite dimensional curvature algebra and hence its holonomy is not a (finite dimensional) Lie group. These results give a positive answer to the following problem formulated by S. S. Chern and Z. Shen in [8] (p. 85): *Is there a Finsler manifold whose holonomy group is not the holonomy group of any Riemannian manifold?*

Section 4 contains construction of further tangent Lie algebras to the holonomy group consisting of of tangent vector fields on the indicatrix, namely the infinitesimal holonomy algebra and the holonomy algebra of a Finsler manifold. Our goal is to make an attempt to find the right notion of the holonomy algebra of Finsler spaces. The holonomy algebra should be the largest Lie algebra such that all its elements are tangent to the holonomy group. In our attempt we are building successively Lie algebras having the tangent properties. We define the *infinitesimal holonomy algebra* by the smallest Lie algebra of vector fields on an indicatrix, containing the curvature vector fields and their horizontal covariant derivatives with respect to the Berwald connection and prove the tangential property of this Lie algebra

to the holonomy group. At the end we introduce the notion of the *holonomy algebra* of a Finsler manifold by all conjugates of infinitesimal holonomy algebras by parallel translations with respect to the Berwald connection. We prove that this holonomy algebra is tangent to the holonomy group. The question of whether the holonomy algebra introduced in this way is the largest Lie algebra, which is tangent to the holonomy group, is still open.
In Section 6 we construct for interesting classes of locally projectively flat Finsler surfaces and manifolds of non-zero constant curvature infinite dimensional subalgebras in the tangent infinitesimal holonomy algebras. From the viewpoint of non-Euclidean geometry the most important Riemann-Finsler manifolds are the projectively flat spaces of constant flag curvature. We will turn our attention to non-Riemannian projectively flat Finsler manifolds of non-zero constant flag curvature. We consider the following classes of locally projectively flat non-Riemannian Finsler manifolds of non-zero constant flag curvature:

1. Randers manifolds,
2. manifolds having a 2-dimensional subspace in the tangent space at some point, on which the Finsler norm is an Euclidean norm,
3. manifolds having a 2-dimensional subspace in the tangent space at some point, on which the Finsler norm and the projective factor are linearly dependent.

The first class consists of positively complete Finsler manifolds of negative curvature, the second class contains a large family of (not necessarily complete) Finsler manifolds of negative curvature, and the third class contains a large family of not necessarily complete Finsler manifolds of positive curvature. The metrics belonging to these classes can be considered as (local) generalizations of a one-parameter family of complete Finsler manifolds of positive curvature defined on S^2 by R. Bryant in [Br1], [Br2] and on S^n by Z. Shen in [37], Example 7.1. We prove that the holonomy group of Finsler manifolds belonging to these classes and satisfying some additional technical assumption is infinite dimensional. In Section 7 we are investigating the holonomy group of an arbitrary locally projectively flat Finsler manifolds of constant curvature. Our aim is to characterize all locally projectively flat Finsler manifolds with finite dimensional holonomy group. To obtain such a characterization, we will investigate the dimension of the infinitesimal holonomy algebra. We prove that if $(M, \mathcal{F})$ is a non-Riemannian locally projectively flat Finsler manifolds of nonzero constant curvature, then its infinitesimal holonomy algebra is infinite dimensional. Using this general result and the tangent property of the infinitesimal holonomy algebra we obtain the characterization: *The holonomy group of a locally projectively flat Finsler manifold of constant curvature is finite dimensional if and only if it is a Riemannian manifold or a flat Finsler manifold.*
Section 8 is devoted to show that the topological closure of the holonomy group of a certain class of simply connected, projectively flat Finsler 2-manifolds of constant curvature is not a finite dimensional Lie group, and we prove that its topological closure is the connected component of the full diffeomorphism group of the circle. Until now, perhaps because of technical difficulties, not a single infinite dimensional Finsler holonomy group has been described. In this paper we provide the first such a description. This class of Finsler 2-manifolds contains the positively complete standard Funk plane of constant negative curvature (positively complete standard Funk plane), and the complete irreversible Bryant-Shen-spheres of constant positive curvature ([37], [5]). We obtain that for every simply connected Finsler 2-manifold the topological closure of the holonomy group is a subgroup of $\mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$. That means that in the examples mentioned above, the closed holonomy group is maximal. In the proof we use our constructive method developed in Section 6 for the study of Lie algebras of vector fields on the indicatrix, which are tangent to the holonomy group.

2 Preliminaries

2.1 Spray manifolds

A *spray* on a manifold M is a smooth vector field $\mathcal{S}$ on $\hat{T}M := TM\setminus\{0\}$ expressed in a standard coordinate system (x^i, y^i) on TM as

$$\mathcal{S} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}, \tag{1}$$

where the functions $G^i(x,y)$ of local coordinates (x^i, y^i) on TM satisfy

$$G^i(x,\lambda y) = \lambda^2 G^i(x,y), \quad \lambda > 0. \tag{2}$$

A manifold M with a spray $\mathcal{S}$ is called a *spray manifold* $(M,\mathcal{S})$, cf. [34], Chapter 4.
A curve $c(t)$ is called *geodesic* of the spray manifold $(M,\mathcal{S})$ if its coordinate functions $c^i(t)$ satisfy the system of 2nd order ordinary differential equations

$$\ddot{c}^i(t) + 2G^i(c(t),\dot{c}(t)) = 0, \tag{3}$$

where the functions $G^i(x,y)$ are called the *geodesic coefficients* the spray manifold $(M,\mathcal{S})$.

2.1.1 Horizontal distribution, covariant derivative and curvature

Let (TM,π,M) and (TTM,τ,TM) be the first and the second tangent bundle of the manifold M, respectively, and let $\mathcal{V}TM\subset TTM$ be the (integrable) vertical distribution on TM given by $\mathcal{V}_yTM := \operatorname{Ker}\pi_{*,y}$. The *horizontal distribution* $\mathcal{H}TM\subset TTM$ associated to the spray manifold $(M,\mathcal{S})$ is the image of the horizontal lift which is a vector space isomorphism $l_y\colon T_xM \to \mathcal{H}_yTM$ for every $x\in M$ and $y\in T_xM$ defined by

$$l_y\Big(\frac{\partial}{\partial x^i}\Big) = \frac{\partial}{\partial x^i} - G^k_i(x,y)\frac{\partial}{\partial y^k}, \quad \text{where} \quad G^i_j = \frac{\partial G^i}{\partial y^j} \tag{4}$$

in the coordinate system (x^i, y^i) of TM. The horizontal distribution is complementary to the vertical distribution, hence we have the decomposition $T_yTM = \mathcal{H}_yTM\oplus\mathcal{V}_yTM$. The projectors corresponding to this decomposition will be denoted by $h : TTM \to \mathcal{H}TM$ and $v : TTM \to \mathcal{V}TM$.
The vertical distribution over the slit tangent bundle $\hat{T}M = TM\setminus\{0\}$ will be denoted by $(\hat{\mathcal{V}}TM,\tau,\hat{T}M)$ and the pull-back bundle of $(\hat{T}M,\pi,M)$ corresponding to the map $\pi : TM \to M$ by $(\pi^*TM,\bar{\pi},\hat{T}M)$. Clearly, the mapping

$$(x,y,\xi^i\frac{\partial}{\partial y^i}) \mapsto (x,y,\xi^i\frac{\partial}{\partial x^i}) :\ \hat{\mathcal{V}}TM \to \pi^*TM \tag{5}$$

is a canonical bundle isomorphism. In the following we will use the isomorphism (5) for the identification of these bundles.
Let $\mathfrak{X}^\infty(M)$ be the vector space of smooth vector fields on the manifold M and $\hat{\mathfrak{X}}^\infty(TM)$ the vector space of smooth sections of the bundle $(\hat{\mathcal{V}}TM,\tau,\hat{T}M)$. The *horizontal covariant derivative* of a section $\xi\in\hat{\mathfrak{X}}^\infty(TM)$ by a vector field $X\in\mathfrak{X}^\infty(M)$ is given by

$$\nabla_X\xi := [h(X),\xi].$$

The horizontal covariant derivative of $\xi(x,y) = \xi^i(x,y)\frac{\partial}{\partial y^i}$ by $X(x) = X^i(x)\frac{\partial}{\partial x^i}$ can be expressed as

$$\nabla_X\xi = \Big(\frac{\partial\xi^i(x,y)}{\partial x^j} - G^k_j(x,y)\frac{\partial\xi^i(x,y)}{\partial y^k} + G^i_{jk}(x,y)\xi^k(x,y)\Big)\, X^j\frac{\partial}{\partial y^i}, \tag{6}$$

where $G^i_{jk} := \frac{\partial G^i_j}{\partial y^k}$.
Defining the horizontal covariant derivative

$$\nabla_X \phi = \Big(\frac{\partial \phi}{\partial x^j} - G^k_j(x,y) \frac{\partial \phi(x,y)}{\partial y^k} \Big) X^j$$

of a smooth function $\phi : \hat{T}M \to \mathbb{R}$, the horizontal covariant derivation (6) can be extended to sections of the tensor bundle over $(\pi^* TM, \pi, \hat{T}M)$, using the canonical bundle isomorphism (5).
The *curvature tensor* field

$$K_{(x,y)}(X,Y) := v\big[X^h, Y^h\big], \qquad X, Y \in T_x M. \tag{7}$$

on the pull-back bundle $(\pi^* TM, \pi, \hat{T}M)$ of the spray manifold $(M, \mathcal{S})$ in a local coordinate system (x^i, y^i) of TM is given by

$$K_{(x,y)} = K^i_{jk}(x,y) dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^i},$$

where

$$K^i_{jk}(x,y) = \frac{\partial G^i_j(x,y)}{\partial x^k} - \frac{\partial G^i_k(x,y)}{\partial x^j} + G^m_j(x,y) G^i_{km}(x,y) - G^m_k(x,y) G^i_{jm}(x,y). \tag{8}$$

The curvature tensor field characterizes the integrability of the horizontal distribution. Namely, if the horizontal distribution $\mathcal{H}TM$ is integrable, then the curvature is identically zero.

2.1.2 Parallel translation

For a spray manifold $(M, \mathcal{S})$ the *parallel vector fields* $X(t) = X^i(t) \frac{\partial}{\partial x^i}$ along a curve$c(t)$ are defined by the solutions of the differential equation

$$D_{\dot{c}} X(t) := \Big(\frac{dX^i(t)}{dt} + G^i_j(c(t), X(t)) \dot{c}^j(t) \Big) \frac{\partial}{\partial x^i} = 0. \tag{9}$$

Using the relations (2) and Euler theorem on homogeneous functions we see that the functions $G^i_j(x,y)$ are positive homogeneous of first order with respect to the variable y, and hence $D_{\dot{c}}(\lambda X(t)) = \lambda D_{\dot{c}} X(t)$ for any $\lambda \geq 0$. The differential equation (9) can be expressed by the horizontal covariant derivative (6) as follows: a vector field $X(t) = X^i(t) \frac{\partial}{\partial x^i}$ along a curve $c(t)$ is parallel if it satisfies the equation

$$\nabla_{\dot{c}} X(t) = \Big(\frac{dX^i(t)}{dt} + G^i_j(c(t), X(t)) \dot{c}^j(t) \Big) \frac{\partial}{\partial x^i} = 0. \tag{10}$$

Clearly, for any $X_0 \in T_{c(0)} M$ there is a unique parallel vector field $X(t)$ along the curve c such that $X_0 = X(0)$. Moreover, if $X(t)$ is a parallel vector field along c, then $\lambda X(t)$ is also parallel along c for any $\lambda \geq 0$. Then the *homogeneous (nonlinear) parallel translation*

$$\tau_c : T_{c(0)} M \to T_{c(1)} M \quad \text{along a curve} \quad c(t)$$

of the spray manifold $(M, \mathcal{S})$ is defined by the positive homogeneous map $\tau_c : X_0 \mapsto X_1$ given by the value $X_1 = X(1)$ at $t = 1$ of the parallel vector field with initial value $X(0) = X_0$.
Since the parallel translation of a spray manifold $(M, \mathcal{S})$ is determined by its horizontal distribution $\mathcal{H}TM \subset TTM$, a spray manifold can be considered as a particular case of a fibered manifold equipped with an Ehresmann connection, (cf. [9]). An Ehresmann connection of a fibered manifold is given by a horizontal distribution, which is complement

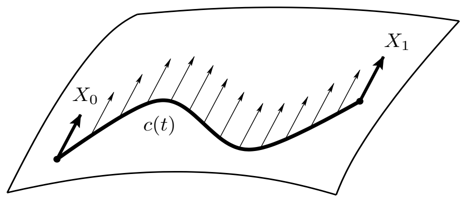


Figure 1: Parallel translation

to the vertical distribution consisting of the tangent spaces of the fibers. For a spray manifold the fibered manifold is the tangent bundle of M and the horizontal distribution determined by the horizontal lift $l_y \colon T_xM \to \mathcal{H}_yTM$ expressed by equation (4).
The parallel translation can be introduced a very nice geometrical way with the help of the notion of horizontal distribution. Namely, we call a curve in TM horizontal if the tangent vectors of this curve are contained in the horizontal distribution $\mathcal{H}TM \subset TTM$. Let now $c(t)$ be a curve in the manifold M joining the points p and q. The *horizontal lift* $c^h(t) = (c(t), X^i(t)\frac{\partial}{\partial x^i})$ of $c(t)$ is the curve $c^h(t)$ in TM defined by the properties that $c^h(t)$ projects on $c(t)$ and $c^h(t)$ is horizontal that is $\dot{c}^h(t) \in H_{c(t)}$. This means according to equation (4) that

$$\dot{c}^i(t)\frac{\partial}{\partial x^i} + \frac{d}{dt}X^i(t)\frac{\partial}{\partial y^i} = \left(\frac{\partial}{\partial x^i} - G^k_i(x,y)\frac{\partial}{\partial y^k}\right)\dot{c}^i(t),$$

i.e. the tangent vector of the lifted curve $c^h(t)$ is the horizontal lift of the tangent vector $\dot{c}^i(t)\frac{\partial}{\partial x^i}$ of $c(t)$. It follows that a vector field $X(t)$ along a curve $c(t)$ is parallel if and only if it is a solution of the differential equation

$$\frac{d}{dt}\left(c(t), X^i(t)\frac{\partial}{\partial x^i}\right) = l_{X(t)}(\dot{c}(t)), \tag{11}$$

or equivalently $X(t)$ satisfies the differential equation (9). Hence the parallel translation along a curve $c(t)$ joining the points p and q is the map $\tau_c : T_pM \to T_qM$ determined by the intersection points of the horizontal lifts of the curve $c(t)$ with the tangent spaces T_p and T_q. The construction can be illustrated by the following figure:

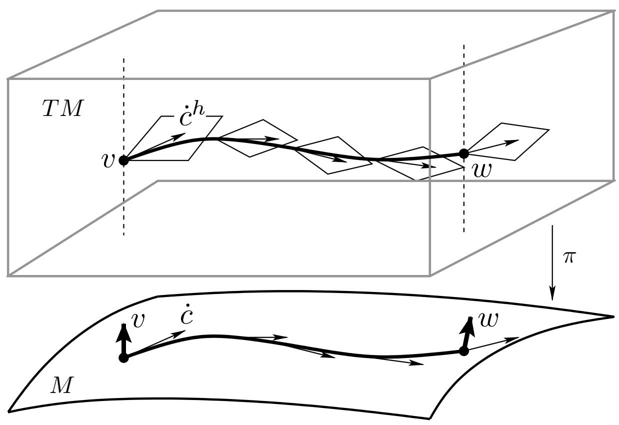


Figure 2: Geometric construction of the parallel translation

2.1.3 Totally geodesic and auto-parallel submanifolds

A submanifold $\bar{M}$ in a spray manifold $(M, \mathcal{S})$ is called *totally geodesic* if any geodesic of $(M, \mathcal{S})$ which is tangent to $\bar{M}$ at some point is contained in $\bar{M}$.

A totally geodesic submanifold $\bar{M}$ of $(M, \mathcal{S})$ is called *auto-parallel* if the homogeneous (nonlinear) parallel translations $\tau_c : T_{c(0)}M \to T_{c(1)}M$ along curves in the submanifold $\bar{M}$ leave invariant the tangent bundle $T\bar{M}$ and for every $\xi \in \hat{\mathfrak{X}}^\infty(T\bar{M})$ the horizontal Berwald covariant derivative $\nabla_X \xi$ belongs to $\hat{\mathfrak{X}}^\infty(T\bar{M})$.

Let $X, Y \in T_xM$ be tangent vectors at $x \in M$ and let K denote the curvature tensor of $(M, \mathcal{S})$, (cf. equation (8)). The mapping $y \to K(X,Y)(x,y) : T_xM \mapsto T_xM$ is called *curvature vector field at* x of the spray manifold $(M, \mathcal{S})$.

Lemma 1 *Let $\bar{M}$ be a totally geodesic submanifold in a spray manifold $(M, \mathcal{S})$. The following assertions hold:*

(a) the spray $\mathcal{S}$ induces a spray $\bar{\mathcal{S}}$ on the submanifold $\bar{M}$,

(b) $\bar{M}$ is an auto-parallel submanifold,

Proof. Assume that the manifolds $\bar{M}$ and M are k, respectively $n = k + p$ dimensional. Let $(x^1, \dots, x^k, x^{k+1}, \dots, x^n)$ be an adapted coordinate system, i. e. the submanifold $\bar{M}$ is locally given by the equations $x^{k+1} = \dots = x^n = 0$. We denote the indices running on the values $\{1, \dots, k\}$ or $\{k+1, \dots, n\}$ by α, β, γ or σ, τ, respectively. The differential equation (3) of geodesics yields that the geodesic coefficients $G^\sigma(x,y)$ satisfy

$$G^\sigma(x^1, \dots, x^k, 0, \dots, 0; y^1, \dots, y^k, 0, \dots, 0) = 0$$

identically, hence their derivatives with respect to $y^1, \dots, y^k$ are also vanishing. It follows that $G^\sigma_\alpha = 0$ and $G^\sigma_{\alpha\,\beta} = 0$ at any $(x^1, \dots, x^k, 0, \dots, 0; y^1, \dots, y^k, 0, \dots, 0)$. Hence the induced spray $\bar{\mathcal{S}}$ on $\bar{M}$ is defined by the geodesic coefficients

$$\bar{G}^\beta(x^1, \dots, x^k; y^1, \dots, y^k) = G^\beta(x^1, \dots, x^k, 0, \dots, 0; y^1, \dots, y^k, 0, \dots, 0). \tag{12}$$

The homogeneous (nonlinear) parallel translation $\tau_c : T_{c(0)}M \to T_{c(1)}M$ along curves in the submanifold $\bar{M}$ and the horizontal covariant derivative on $\bar{M}$ with respect to the spray $\mathcal{S}$ coincide with the translation and the horizontal covariant derivative on $\bar{M}$ with respect to the spray $\bar{\mathcal{S}}$. Hence the assertions are true. ∎

2.1.4 Holonomy

The notion of the holonomy group of an Ehresmann connection, or particularly of a spray manifold can be introduced in a very natural way: it is the group generated by parallel translations along loops with respect to the associated connection, cf. [41], [18], pp. 82–86. The holonomy properties of a spray manifold depends essentially on its curvature properties. This can be easily understood by considering the geometric construction of the parallel translation.

Let $(M, \mathcal{S})$ be a spray manifold, and let us assume that M is connected. We choose a fixed base point $p \in M$. For each closed piecewise smooth curve $c : [0,1] \to M$ through p the parallel translation $\tau_c : T_pM \to T_pM$ along the curve $c : [0,1] \to M$ is a diffeomorphism of the tangent space T_pM. All these diffeomorphisms form together the holonomy group at the point p, a subgroup of the diffeomorphism group of T_pM. Clearly, the holonomy group depends on the base point p only up to conjugation, and therefore the holonomy groups at different points of M are isomorphic.

• Case $K \equiv 0$. Let us consider a point $p \in M$, a tangent vector $v \in T_pM$ at p, and an

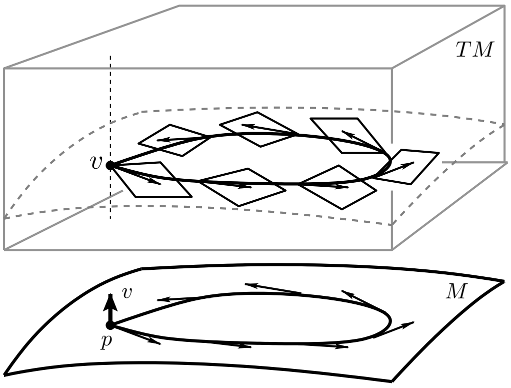


Figure 3: Trivial holonomy: $R \equiv 0$

arbitrary closed curve c on M starting from p. If the curvature identically zero then the horizontal distribution is integrable. Therefore TTM has a horizontal foliation, that is for every $v \in T_pM$ there is a unique n-dimensional submanifold $\mathcal{H}_v$ in TM such that $\mathcal{H}_v \cap T_pM = \{v\}$ and the tangent spaces are horizontal. On the Figure 3 the level surface corresponding to $v \in T_pM$ is represented by dashed lines in tangent manifold TM.

The horizontal lift $c^h = l \circ c$ of the curve c starts at v and (because the integrability condition) stays on the horizontal submanifold $\mathcal{H}_v$. Its endpoint is an element of $\mathcal{H}_v$ and also an element of T_pM that is v. Consequently we can obtain that $c^h(0) = c^h(1) = v$, and the holonomy is trivial.

- Case $K \not\equiv 0$. Although the figure describing this case looks similar to that of Figure 3, the situation is quite different. The horizontal distribution is non-integrable and therefore there is no horizontal foliation. Let us consider a point $p \in M$ and a vector $v \in T_pM$. The smallest integrable distribution $\mathcal{N}_v$ at v containing the horizontal distribution is at least $(n+1)$-dimensional. The reachable sets in T_pM is at least 1-dimensional. In particular $\{v\} \subsetneq \mathcal{N}_v \cap T_pM$ and there are other elements T_pM reachable from v. That is there are elements $w \in T_pM$ and a horizontal curve $c^h \in \mathcal{N}_v$ such that $c^h(0) = v$, $c^h(1) = w$. Considering the projection of c^h we can obtain a curve $c := \pi \circ c^h$ such that $c(0) = c(1) = p$ and $\tau_c(v) \neq v$. Consequently we obtained that the holonomy is nontrivial. (Cf. [34], Remark 8.1.3.)

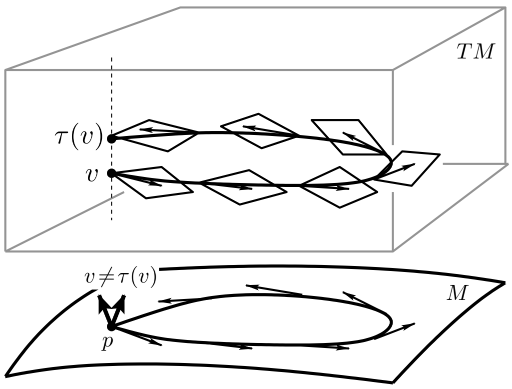


Figure 4: Nontrivial holonomy: $R \not\equiv 0$

2.2 Finsler manifolds

2.2.1 Finsler metric, its associated spray and parallel translation

A *Minkowski functional* on a vector space V is a continuous function $\mathcal{F}$, positively homogeneous of degree two, i.e. $\mathcal{F}(\lambda y) = \lambda^2 \mathcal{F}(y)$ if $\lambda > 0$, smooth on $\hat{V} := V \setminus \{0\}$, and for any $y \in \hat{V}$ the symmetric bilinear form $g_y \colon V \times V \to \mathbb{R}$ defined by

$$g_y \colon (u, v) \ \mapsto \ g_{ij}(y) u^i v^j = \frac{1}{2} \frac{\partial^2 \mathcal{F}(y + su + tv)}{\partial s \, \partial t} \Big|_{t=s=0}$$

is non-degenerate. If g_y is positive definite for any $y \in \hat{V}$ then $\mathcal{F}$ is said positive definite and $(V, \mathcal{F})$ is called *positive definite Minkowski space*. A Minkowski functional $\mathcal{F}$ is called *semi-Euclidean* if there exists a symmetric bilinear form $\langle \, , \, \rangle$ on V such that $g_y(u, v) = \langle u, v \rangle$ for any $y \in \hat{V}$ and $u, v \in V$. A semi-Euclidean positive definite Minkowski functional is called

Euclidean.

A *Finsler manifold* is a pair $(M, \mathcal{F})$ of an n-manifold M and a function $\mathcal{F}\colon TM \to \mathbb{R}$ (called *Finsler metric*, cf. [34]) defined on the tangent bundle of M, which is smooth on $\hat{T}M := TM \setminus \{0\}$ and its restriction $\mathcal{F}_x = \mathcal{F}|_{T_xM}$ is a Minkowski functional on T_xM for all $x \in M$. If the Minkowski functional $\mathcal{F}_x$ is positive definite on T_xM for all $x \in M$ then $(M, \mathcal{F})$ is called *positive definite* Finsler manifold. A point $x \in M$ is called *(semi-)Riemannian* if the Minkowski functional $\mathcal{F}_x$ is (semi-)Euclidean.

We remark that in many applications the metric $\mathcal{F}$ is defined and smooth only on an open cone $\mathcal{C}M \subset TM\setminus\{0\}$, where $\mathcal{C}M = \cup_{x\in M}\mathcal{C}_xM$ is a fiber bundle over M such that each $\mathcal{C}_xM$ is an open cone in $T_xM\setminus\{0\}$. In such case $(M, \mathcal{F})$ is called *singular* (or *non y-global*) Finsler space (cf. [34]).

The symmetric bilinear form

$$g_{x,y}\colon (u,v) \;\mapsto\; g_{ij}(x,y)u^iv^j = \frac{1}{2}\frac{\partial^2\mathcal{F}_x^2(y+su+tv)}{\partial s\,\partial t}\Big|_{t=s=0}, \qquad u,v \in T_xM$$

is called the metric tensor of the Finsler manifold $(M, \mathcal{F})$. The Finsler function is called *absolutely homogeneous* at $x \in M$, if $\mathcal{F}_x(\lambda y) = |\lambda|\mathcal{F}_x(y)$ for all $\lambda \in \mathbb{R}$. If $\mathcal{F}$ is absolutely homogeneous at every $x \in M$, then the Finsler manifold $(M, \mathcal{F})$ is *reversible.*

The simplest non-Riemannian Finsler metrics are the Randers metrics firstly studied by G. Randers in [33]. A Finsler manifold $(M, \mathcal{F})$ is called *Randers manifold* if the Finsler metric $\mathcal{F}$ can be expressed in the form $F(x,y) = a_x(y) + b_x(y)$, where $a_x(y) = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $b_x(y) = b_i(x)y^i$ is a nowhere zero 1-form.

The *canonical spray* of a Finsler manifold $(M, \mathcal{F})$ is locally given by $\mathcal{S} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where

$$G^i(x,y) := \frac{1}{4}g^{il}(x,y)\Big(2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\Big)y^jy^k. \tag{13}$$

are the *geodesic coefficients.* The *geodesics* of a Finsler manifold $(M, \mathcal{F})$ are the geodesics of the canonical spray of $(M, \mathcal{F})$ determined by the differential equations (3). The horizontal covariant derivative with respect to the spray associated to the Finsler manifold $(M, \mathcal{F})$ is called the *horizontal Berwald covariant derivative*, cf. equation (6). The corresponding homogeneous (nonlinear) parallel translation $\tau_c : T_{c(0)}M \to T_{c(1)}M$ along a curve $c(t)$, given by equations (9) and (10), is called the *canonical homogeneous (nonlinear) parallel translation* of the Finsler manifold $(M, \mathcal{F})$. Since the geodesic coefficients $G^i(x,y)$ are differentiable functions on the slit tangent bundle $\hat{T}M = TM \setminus \{0\}$ the canonical homogeneous (nonlinear) parallel translation $\tau_c : T_{c(0)}M \to T_{c(1)}M$ induces differentiable maps between the slit tangent spaces $T_{c(0)}M \setminus \{0\}$ and $T_{c(1)}M \setminus \{0\}$.

2.2.2 Curvature

The *Riemannian curvature tensor* field of the Finsler manifold $(M, \mathcal{F})$ is the *curvature tensor* field $v\big[X^h, Y^h\big]$ of the canonical spray manifold of $(M, \mathcal{F})$ is defined on the pull-back bundle $(\pi^*TM, \bar{\pi}, \hat{T}M)$, (cf. equation (7)). If $\mathcal{H}TM$ is integrable, then the Riemannian curvature is identically zero. According to equation (8) the expression of the Riemannian curvature tensor $R_{(x,y)} = R^i_{jk}(x,y)dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^i}$ is

$$R^i_{jk}(x,y) = \frac{\partial G^i_j(x,y)}{\partial x^k} - \frac{\partial G^i_k(x,y)}{\partial x^j} + G^m_j(x,y)G^i_{km}(x,y) - G^m_k(x,y)G^i_{jm}(x,y)$$

in a a local coordinate system. The manifold $(M, \mathcal{F})$ has constant flag curvature $\lambda \in \mathbb{R}$, if for any $x \in M$ the local expression of the Riemannian curvature is

$$R^i_{jk}(x,y) = \lambda\left(\delta^i_k g_{jm}(x,y)y^m - \delta^i_j g_{km}(x,y)y^m\right). \tag{14}$$

In this case the flag curvature of the Finsler manifold (cf. [8], Section 2.1 pp. 43-46) does not depend either on the point or on the 2-flag.

The *Berwald curvature tensor* field $B_{(x,y)} = B^i_{jkl}(x,y)dx^j \otimes dx^k \otimes dx^l \otimes \frac{\partial}{\partial x^i}$ is

$$B^i_{jkl}(x,y) = \frac{\partial G^i_{jk}(x,y)}{\partial y^l} = \frac{\partial^3 G^i(x,y)}{\partial y^j \partial y^k \partial y^l}. \tag{15}$$

The *mean Berwald curvature tensor* field $E_{(x,y)} = E_{jk}(x,y)dx^j \otimes dx^k$ is the trace

$$E_{jk}(x,y) = B^l_{jkl}(x,y) = \frac{\partial^3 G^l(x,y)}{\partial y^j \partial y^k \partial y^l}. \tag{16}$$

The *Landsberg curvature tensor* field $L_{(x,y)} = L^i_{jkl}(x,y)dx^j \otimes dx^k \otimes dx^l \otimes \frac{\partial}{\partial x^i}$ is

$$L_{(x,y)}(u,v,w) = g_{(x,y)}\left(\nabla_w B_{(x,y)}(u,v,w), y\right), \quad u,v,w \in T_xM.$$

According to Lemma 6.2.2, equation (6.30), p. 85 in [34], one has for $u,v,w \in T_xM$

$$\nabla_w g_{(x,y)}(u,v) = -2L_{(x,y)}(u,v,w).$$

Lemma 2 *The horizontal Berwald covariant derivative of the tensor field*

$$Q_{(x,y)} = \left(\delta^i_j g_{km}(x,y)y^m - \delta^i_k g_{jm}(x,y)y^m\right) dx^j \otimes dx^k \otimes dx^l \otimes \frac{\partial}{\partial x^i}$$

vanishes.

Proof. For any vector field $W \in \mathfrak{X}^\infty(M)$ we have $\nabla_W y = 0$ and $\nabla_W \mathsf{Id}_{TM} = 0$. Moreover, since $L_{(x,y)}(y,v,w) = 0$ (cf. equation 6.28, p. 85 in [34]) we get the assertion. ∎

2.2.3 Projectively flat Finsler manifold

A Finsler manifold $(D, \mathcal{F})$ on an open subset $D \subset \mathbb{R}^n$ is said to be *projectively flat*, if all geodesics of $(D, \mathcal{F})$ are contained in straight lines of the affine space associated to $\mathbb{R}^n$. A Finsler manifold $(M, \mathcal{F})$ is said to be *locally projectively flat*, if for any point in $p \in M$ there exists a local coordinate map $x : U \to \mathbb{R}^n$ of a neighbourhood $U \subset M$ of p such that the Finsler manifold induced by the Finsler function $\mathcal{F}$ on the image $x(U) = D$ is projectively flat. The space $\mathbb{R}^n$ containing D is called to be *projectively related to* $(M, \mathcal{F})$.

Let $(M, \mathcal{F})$ be a locally projectively flat Finsler manifold and $(x^1, \dots, x^n) : U \to D$ a local coordinate map corresponding to canonical coordinates of the space $\mathbb{R}^n$ which is projectively related to $(M, \mathcal{F})$. Then the geodesic coefficients (13) are of the form

$$G^i(x,y) = \mathcal{P}(x,y)y^i, \quad G^i_k = \frac{\partial \mathcal{P}}{\partial y^k}y^i + \mathcal{P}\delta^i_k, \quad G^i_{kl} = \frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^l}y^i + \frac{\partial \mathcal{P}}{\partial y^k}\delta^i_l + \frac{\partial \mathcal{P}}{\partial y^l}\delta^i_k, \tag{17}$$

where $\mathcal{P}$ is a 1-homogeneous function in y, called the *projective factor* of $(M, \mathcal{F})$, (cf. [8], p. 63). Clearly, the intersections of 2-planes of A^n with the image D of the coordinate map $(x^1, \dots, x^n) : U \to D$ are images of totally geodesic submanifolds of $(M, \mathcal{F})$.

Remark 3 *The canonical homogeneous parallel translation $\tau_c : T_{c(0)}M \to T_{c(1)}M$ in a locally projectively flat Finsler manifold $(M, \mathcal{F})$ along curves $c(t)$ contained in the domain of the coordinate system $(x^1, \dots, x^n)$ are linear maps if and only if the projective factor $\mathcal{P}(x,y)$ is a linear function in y. Hence the non-linearity in y of the projective factor implies that the locally projectively flat Finsler manifold is non-Riemannian.*

Projectively flat Randers manifolds with constant flag curvature were classified by Z. Shen in [36]. He proved that any projectively flat Randers manifold $(M, \mathcal{F})$ with non-zero constant flag curvature has negative curvature. These metrics can be normalized by a constant factor so that the curvature is $-\frac{1}{4}$. In this case $(M, \mathcal{F})$ is isometric to the Finsler manifold defined by the metric function

$$\mathcal{F}(x,y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1-|x|^2} \pm \left(\frac{\langle x, y\rangle}{1-|x|^2} + \frac{\langle a, y\rangle}{1+\langle a, x\rangle} \right) \tag{18}$$

on the unit ball $\mathbb{D}^n \subset \mathbb{R}^n$, where $a \in \mathbb{R}^n$ is any constant vector with $|a| < 1$. According to Lemma 8.2.1 in [8], p.155, the projective factor $\mathcal{P}(x, y)$ can be computed by the formula

$$\mathcal{P}(x,y) = \frac{1}{2\mathcal{F}} \frac{\partial \mathcal{F}}{\partial x^i} y^i.$$

An easy calculation yields

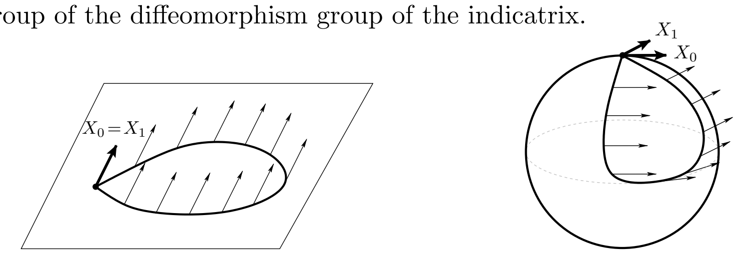
$$\pm \frac{\partial \mathcal{F}}{\partial x^i} y^i = \left(\frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} \pm \langle x, y\rangle}{1-|x|^2} \right)^2 - \left(\frac{\langle a, y\rangle}{1+\langle a, x\rangle} \right)^2,$$

hence

$$\mathcal{P}(x,y) = \frac{1}{2} \left(\frac{\pm\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}{1-|x|^2} - \frac{\langle a, y\rangle}{1+\langle a, x\rangle} \right). \tag{19}$$

2.3 Finsler holonomy represented on the indicatrix bundle

The notion of the holonomy group of a Riemannian or Finslerian manifolds is an adaptation of the corresponding notion of spray manifolds: it is the group generated by parallel translations along loops with respect to the canonical associated connection. However, the parallel translation leaves invariant the indicatrix bundle, the holonomy group can be identified by its action on the indicatrix at the initial point. Hence the holonomy group can be considered as a subgroup of the diffeomorphism group of the indicatrix.



(a) Trivial holonomy: the plane (b) Nontrivial holonomy: sphere

Figure 5: Examples

In the Riemannian case, the holonomy groups have been extensively studied. One of the earliest fundamental results is the theorem of Borel and Lichnerowicz [3] from 1952, claiming that the holonomy group of a simply connected Riemannian manifold is a closed Lie subgroup of the orthogonal group $O(n)$. By now, the complete classification of Riemannian holonomy groups is known.

The holonomy properties of Finsler spaces is, however, essentially different from the Riemannian one, and it is far from being well understood. Compared to the Riemannian case, only few results are known. The main difficulty comes from the fact that in the general case the canonical connection of a Finsler manifold is neither linear nor metrical (that is the parallel translation is not necessarily preserving the metric). Only much weaker properties are fulfilled: instead of the linearity it is only 1−homogeneous, and instead of the metrical property it is preserving only the norm function. Nonetheless these properties allow us to consider the parallel translations as maps between the indicatrices and therefore the holonomy group as a subgroup of the diffeomorphism group of the indicatrix.

Let $(M,\mathcal{F})$ be an n-dimensional Finsler manifold. The *indicatrix* $\mathfrak{I}_xM$ at $x \in M$ is a hypersurface of T_xM defined by

$$\mathfrak{I}_xM := \{y \in T_xM;\ \mathcal{F}(y) = \pm 1\}.$$

If the Finsler manifold $(M,\mathcal{F})$ is positive definite then the indicatrix $\mathfrak{I}_xM$ is a compact hypersurface in the tangent space T_xM, diffeomorphic to the standard $(n-1)$-dimensional sphere.
In the sequel $(\mathfrak{I}M,\pi,M)$ will denote the *indicatrix bundle* of $(M,\mathcal{F})$ and $i : \mathfrak{I}M \hookrightarrow TM$ the natural embedding of the indicatrix bundle into the tangent bundle (TM,π,M).

The *parallel translation* $\tau_c : T_{c(0)}M \to T_{c(1)}M$ along a curve $c : [0,1] \to \mathbb{R}$ on a Finsler manifold $(M,\mathcal{F})$ is defined by the parallel translation of the associated spray manifold (cf. Section 2.1.2). It is determined by vector fields $X(t)$ along $c(t)$ which are solutions of the differential equation (10). Since $\tau_c : T_{c(0)}M \to T_{c(1)}M$ is a differentiable map between the slit tangent spaces $\hat{T}_{c(0)}M$ and $\hat{T}_{c(1)}M$ preserves the value of the Finsler function, it induces a map

$$\tau_c^{\mathfrak{I}} : \mathfrak{I}_{c(0)}M \longrightarrow \mathfrak{I}_{c(1)}M \tag{20}$$

between the indicatrices. Since the parallel translation is 1-homogeneous, the parallel translation τ_c is entirely characterized by the map $\tau_c^{\mathfrak{I}}$: we have $\tau_C(0) = 0$ and for every non-zero vector $v \in T_{c(0)}M$ we have

$$\tau_c(v) = |v| \cdot \tau_c^{\mathfrak{I}}\left(\frac{v}{|v|}\right).$$

It follows from these observations that the holonomy group $\mathsf{Hol}(x)$ of the spray manifold associated to a Finsler manifold $(M,\mathcal{F})$ (cf. Section 2.1.4) is uniquely determined by its action on the indicatrix in the tangent space T_xM at the point x. Hence we can formulate

Definition 4 *The* holonomy group $\mathsf{Hol}(x)$ *of a Finsler space* $(M,\mathcal{F})$ *at* $x \in M$ *is the subgroup of the group of diffeomorphisms* $\mathsf{Diff}(\mathfrak{I}_xM)$ *of the indicatrix* $\mathfrak{I}_xM$ *determined by parallel translation of* $\mathfrak{I}_xM$ *along piece-wise differentiable closed curves initiated at the point* $x \in M$.

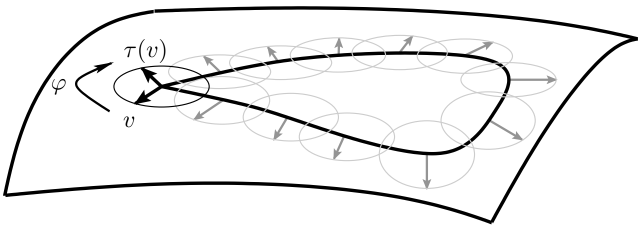


Figure 6: Holonomy transformation induced on the indicatrix

We note that the holonomy group $\mathsf{Hol}(x)$ is a topological subgroup of the regular infinite dimensional Lie group $\mathsf{Diff}(\mathfrak{I}_xM)$, but its differentiable structure is not known in general. ($\mathsf{Diff}(\mathfrak{I}_xM)$ denotes the group of all C^∞-diffeomorphism of $\mathfrak{I}_xM$ with the C^∞-topology.)

3 Diffeomorphism groups and their tangent algebras

The group $\mathsf{Diff}(M)$ of all smooth diffeomorphisms of a differentiable manifold M is a regular infinite dimensional Lie group modeled on the vector space $\mathfrak{X}_c(M)$ of smooth vector fields on M with compact support. The Lie algebra of the infinite dimensional Lie group $\mathsf{Diff}(M)$ is the vector space $\mathfrak{X}_c(M)$, equipped with the negative of the usual Lie bracket, (c.f. A. Kriegl and P. W. Michor [21], Section 43.1, p. 454-456).

Here we discuss the tangential properties of Lie algebras of vector fields to an abstract subgroup of the diffeomorphism group of a manifold. The results of this section will be applied in the following to the investigation of tangent Lie algebras of the holonomy subgroup of the diffeomorphism group of an indicatrix $\mathfrak{I}_x M$ and to the fibred holonomy subgroup of the diffeomorphism group of the indicatrix bundle $\mathfrak{I}(M)$.

Let M be a C^∞ manifold, let $\mathcal{G}$ be a (not necessarily differentiable) subgroup of the diffeomorphism group $\mathsf{Diff}^\infty(M)$ and let $\mathfrak{X}^\infty(M)$ be the Lie algebra of smooth vector fields on M.

Definition 5 A vector field $X \in \mathfrak{X}^\infty(M)$ is called *tangent* to the subgroup $\mathcal{G}$ of $\mathsf{Diff}^\infty(M)$, if there exists a $\mathcal{C}^1$-differentiable 1-parameter family $\{\phi_t \in \mathcal{G}\}_{t\in(-\varepsilon,\varepsilon)}$ of diffeomorphisms of M such that $\phi_0 = \mathsf{Id}$ and $\frac{\partial \phi_t}{\partial t}\big|_{t=0} = X$. A Lie subalgebra $\mathfrak{g}$ of $\mathfrak{X}^\infty(M)$ is called *tangent* to $\mathcal{G}$, if all elements of $\mathfrak{g}$ are tangent vector fields to $\mathcal{G}$.

Unfortunately, it is not true, that tangent vector fields to the group $\mathcal{G}$ generate a tangent Lie algebra to $\mathcal{G}$. This is why we have to introduce a stronger tangential property in Definition 7.

Definition 6 A $\mathcal{C}^\infty$-differentiable k-parameter family $\{\phi_{(t_1,\dots,t_k)} \in \mathsf{Diff}^\infty(M)\}_{t_i\in(-\varepsilon,\varepsilon)}$ of diffeomorphisms of M is called a *commutator-like family* if it satisfies the equations

$$\phi_{(t_1,\dots,t_k)} = \mathsf{Id}, \quad \text{whenever} \quad t_j = 0 \quad \text{for some} \quad 1 \le j \le k.$$

We remark, that any $\mathcal{C}^1$-differentiable 1-parameter family $\{\phi_t \in \mathcal{G}\}_{t\in(-\varepsilon,\varepsilon)}$ of diffeomorphisms of M with $\phi_0 = \mathsf{Id}$ is a commutator-like family. Moreover, the commutators of commutator-like families are commutator-like, and the inverse of commutator-like families are commutator-like.

Definition 7 A vector field $X \in \mathfrak{X}^\infty(M)$ is called *strongly tangent* to the subgroup $\mathcal{G}$ of $\mathsf{Diff}^\infty(M)$, if there exists a commutator-like family $\{\phi_{(t_1,\dots,t_k)} \in \mathsf{Diff}^\infty(M)\}_{t_i\in(-\varepsilon,\varepsilon)}$ of diffeomorphisms satisfying the conditions

(A) $\phi_{(t_1,\dots,t_k)} \in \mathcal{G}$ for all $t_i \in (-\varepsilon,\varepsilon)$, $1 \le i \le k$,

(B) $\frac{\partial^k \phi_{(t_1,\dots,t_k)}}{\partial t_1 \cdots \partial t_k}\big|_{(0,\dots,0)} = X$.

It follows from the commutator-like property that $\frac{\partial^k \phi_{(t_1,\dots,t_k)}}{\partial t_1 \cdots \partial t_k}\big|_{(0,\dots,0)}$ is the first non-necessarily vanishing derivative of the diffeomorphism family $\{\phi_{(t_1,\dots,t_k)}\}$ at any point $x \in M$, and therefore it determines a vector field. On the other hand, by reparametrizing the commutator like family of diffeomorphism, it can be shown that if a vector field is strongly tangent to a group $\mathcal{G}$, then it is also tangent to $\mathcal{G}$. Moreover, we have the following

Theorem 8 *Let $\mathcal{V}$ be a set of vector fields strongly tangent to the subgroup $\mathcal{G}$ of $\mathsf{Diff}^\infty(M)$. The Lie subalgebra $\mathfrak{v}$ of $\mathfrak{X}^\infty(M)$ generated by $\mathcal{V}$ is tangent to $\mathcal{G}$.*

Proof. First, we investigate some properties of vector fields strongly tangent to the group $\mathcal{G}$.

Lemma 9 *Let $\{\psi_{(t_1,\dots,t_h)} \in \mathsf{Diff}^\infty(U)\}_{t_i\in(-\varepsilon,\varepsilon)}$ be a $\mathcal{C}^\infty$-differentiable h-parameter commutator-like family of (local) diffeomorphisms on a neighbourhood $U \subset \mathbb{R}^n$. Then*

(i) $$\frac{\partial^{i_1+\dots+i_h}\psi_{(t_1,\dots,t_h)}}{\partial t_1^{i_1} \dots \partial t_h^{i_h}}\bigg|_{(0,\dots,0)}(x) = 0, \quad \textit{if} \quad i_p = 0 \quad \textit{for some} \quad 1 \le p \le h;$$

(ii) $\dfrac{\partial^h(\psi_{(t_1,\ldots,t_h)})^{-1}}{\partial t_1\ \ldots\ \partial t_h}\Big|_{(0,\ldots,0)}(x) = -\dfrac{\partial^h\psi_{(t_1,\ldots,t_h)}}{\partial t_1\ \ldots\ \partial t_h}\Big|_{(0,\ldots,0)}(x);$

(iii) $\dfrac{\partial^h\psi_{(t_1,\ldots,t_h)}}{\partial t_1\ \ldots\ \partial t_h}\Big|_{(0,\ldots,0)}(x) = \dfrac{\partial\psi\ \sqrt[h]{t},\ldots,\sqrt[h]{t})}{\partial t}\Big|_{t=0}(x)$

at any point $x \in U$.

Proof. Assertions (i) and (ii) can be obtained by direct computation. It follows from (i) that $\frac{\partial^h\psi_{(t_1,\ldots,t_h)}}{\partial t_1\ \ldots\ \partial t_h}\Big|_{(0,\ldots,0)}(x)$ is the first non-necessarily vanishing derivative of the diffeomorphism family $\{\psi_{(t_1,\ldots,t_h)}\}$ at any point $x \in M$. Using

$$\psi_{(t_1,\ldots,t_k)}(x) = x + t_1\cdots t_k\left(X(x) + \omega(x, t_1, \ldots, t_k)\right),$$

where $\lim\limits_{t_i\to 0}\omega(x, t_1, \ldots, t_k) = 0$ we obtain, that

$$\frac{\partial}{\partial t}\Big|_{t=0}\psi_{(\sqrt[k]{t},\ldots,\sqrt[k]{t})}(x) = \frac{\partial}{\partial t}\Big|_{t=0}\Big(x + t\big(X(x) + \omega(x, \sqrt[k]{t}, \ldots, \sqrt[k]{t})\big)\Big) = X(x),$$

which proves (iii). ∎

We remark that assertion (iii) means that any vector field strongly tangent to $\mathcal{G}$ is tangent to $\mathcal{G}$.
Now, we generalize a well-known relation between the commutator of vector fields and the commutator of their induced flows.

Lemma 10 *Let $\{\phi_{(s_1,\ldots,s_k)}\}$ and $\{\psi_{(t_1,\ldots,t_l)}\}$ be $\mathcal{C}^\infty$-differentiable k-parameter, respectively l-parameter families of (local) diffeomorphisms defined on a neighbourhood $U \subset \mathbb{R}^n$. Assume that $\phi_{(s_1,\ldots,s_k)} = \mathsf{Id}$, respectively $\psi_{(t_1,\ldots,t_l)} = \mathsf{Id}$, if some of their variables is 0. Then the family of (local) diffeomorphisms $[\phi_{(s_1,\ldots,s_k)}, \psi_{(t_1,\ldots,t_l)}]$ defined by the commutator of the group $\mathsf{Diff}^\infty(U)$ fulfills $[\phi_{(s_1,\ldots,s_k)}, \psi_{(t_1,\ldots,t_l)}] = \mathsf{Id}$, if some of its variables equals 0. Moreover*

$$\frac{\partial^{k+l}[\phi_{(s_1\ldots s_k)}, \psi_{(t_1\ldots t_l)}]}{\partial s_1\ \ldots\ \partial s_k\ \partial t_1\ \ldots\ \partial t_l}\Big|_{(0\ldots0;0\ldots0)}(x) = -\left[\frac{\partial^k\phi_{(s_1\ldots s_k)}}{\partial s_1\ \ldots\ \partial s_k}\Big|_{(0\ldots0)}, \frac{\partial^l\psi_{(t_1\ldots t_l)}}{\partial t_1\ \ldots\ \partial t_l}\Big|_{(0\ldots0)}\right](x)$$

at any point $x \in U$.

Proof. The group theoretical commutator $\left[\phi_{(s_1,\ldots,s_k)}, \psi_{(t_1,\ldots,t_l)}\right]$ of the families of diffeomorphisms satisfies $[\phi_{(s_1,\ldots,s_k)}, \psi_{(t_1,\ldots,t_l)}] = \mathsf{Id}$, if some of its variables equals 0. Hence

$$\frac{\partial^{i_1+\ldots+i_k+j_1+\ldots+j_l}[\phi_{(s_1,\ldots,s_k)}, \psi_{(t_1,\ldots,t_l)}]}{\partial s_1^{i_1}\ \ldots\ \partial s_k^{i_k}\partial t_1^{j_1}\ \ldots\ \partial t_l^{i_l}}\Big|_{(0,\ldots,0;0,\ldots,0)} = 0,$$

if $i_p = 0$ or $j_q = 0$ for some index $1 \le p \le k$ or $1 \le q \le l$. The families of diffeomorphisms $\{\phi_{(s_1,\ldots,s_l)}\}$, $\{\psi_{(t_1,\ldots,t_l)}\}$, $\{\phi^{-1}_{(s_1,\ldots,s_l)}\}$ and $\{\psi^{-1}_{(t_1,\ldots,t_l)}\}$ are the constant family Id, if some of their variables equals 0. Hence one has

$$\frac{\partial^{k+l}[\phi_{(s_1\ldots s_k)}, \psi_{(t_1\ldots t_l)}]}{\partial s_1\ \ldots\ \partial s_k\ \partial t_1\ \ldots\ \partial t_l}\Big|_{(0,\ldots,0;\ 0,\ldots,0)}(x) = \tag{21}$$

$$= \frac{\partial^k}{\partial s_1\ldots\partial s_k}\Big|_{(0\ldots0)}\left(\frac{\partial^l\left(\phi^{-1}_{(s_1\ldots s_k)}\circ\psi^{-1}_{(t_1\ldots t_l)}\circ\phi_{(s_1\ldots s_k)}\circ\psi_{(t_1\ldots t_l)}(x)\right)}{\partial t_1\ldots\partial t_l}\Big|_{(0\ldots0)}\right)$$

$$= \frac{\partial^k}{\partial s_1\ldots\partial s_k}\Big|_{(0\ldots0)}\left(d(\phi^{-1}_{(s_1\ldots s_k)})_{\phi_{(s_1\ldots s_k)}(x)}\frac{\partial^l\psi^{-1}_{(t_1\ldots t_l)}}{\partial t_1\ldots\partial t_l}\Bigg|_{(0,\ldots,0)}(\phi_{(s_1\ldots s_k)}(x))\right),$$

where $d\big(\phi^{-1}_{(s_1,\dots,s_k)}\big)_{\phi_{(s_1,\dots,s_k)}(x)}$ denotes the Jacobi operator of the map $\phi^{-1}_{(s_1,\dots,s_k)}$ at the point $\phi_{(s_1,\dots,s_k)}(x)$. Using the fact, that $\{\phi_{(s_1,\dots,s_k)}\}$ is the constant family Id, if some of its variables equals 0, and the relation $d(\phi^{-1}_{(0,\dots,0)})_{\phi_{(s_1,\dots,s_k)}(x)} = \mathsf{Id}$, we obtain that (21) can be written as

$$d\Big(\frac{\partial^k \phi^{-1}_{(s_1\dots s_k)}}{\partial s_1 \dots \partial s_k}\Big|_{(0\dots 0)}\Big)_x \frac{\partial^l \psi^{-1}_{(t_1\dots t_l)}(x)}{\partial t_1 \dots \partial t_l}\Big|_{(0\dots 0)} + d\Big(\frac{\partial^l \psi^{-1}_{(t_1\dots t_l)}}{\partial t_1 \dots \partial t_l}\Big|_{(0\dots 0)}\Big)_x \frac{\partial^k \phi_{(s_1\dots s_k)}(x)}{\partial s_1 \dots \partial s_k}\Big|_{(0,\dots,0)}.$$

According to assertion (ii) of Lemma 9 the last formula gives

$$d\Big(\frac{\partial^k \phi_{(s_1\dots s_k)}}{\partial s_1 \dots \partial s_k}\Big|_{(0\dots 0)}\Big)_x \frac{\partial^l \psi_{(t_1\dots t_l)}(x)}{\partial t_1 \dots \partial t_l}\Big|_{(0\dots 0)} - d\Big(\frac{\partial^l \psi_{(t_1\dots t_l)}}{\partial t_1 \dots \partial t_l}\Big|_{(0\dots 0)}\Big)_x \frac{\partial^k \phi_{(s_1\dots s_k)}(x)}{\partial s_1 \dots \partial s_k}\Big|_{(0\dots 0)},$$

which is the Lie bracket of vector fields

$$\left[\frac{\partial^l \psi_{(t_1,\dots,t_l)}}{\partial t_1 \dots \partial t_l}\Big|_{(0,\dots,0)}, \frac{\partial^k \phi_{(s_1,\dots,s_k)}}{\partial s_1 \dots \partial s_k}\Big|_{(0,\dots,0)}\right] : U \to \mathbb{R}^n.$$

■

The previous lemma gives for 1-parameter families $\{\phi_t \in \mathcal{G}\}_{t\in(-\varepsilon,\varepsilon)}$ and $\{\psi_t \in \mathcal{G}\}_{t\in(-\varepsilon,\varepsilon)}$ of diffeomorphisms of M with $\phi_0 = \psi_0 = \mathsf{Id}$ the relation

$$\frac{\partial^2}{\partial s \partial t}\Big|_{s=0,t=0}[\phi_s, \psi_t] = -\left[\frac{\partial \phi_s}{\partial s}\Big|_{s=0}, \frac{\partial^l \psi_t}{\partial t}\Big|_{t=0}\right]. \tag{22}$$

Lemma 11 *Any Lie subalgebra of $\mathfrak{X}^\infty(M)$ algebraically generated by strongly tangent vector fields to the group $\mathcal{G}$ has a basis consisting of vector fields strongly tangent to the group $\mathcal{G}$.*

Proof. Let $\mathcal{V}$ be a set of strongly tangent vector fields to the group $\mathcal{G}$ and $\mathfrak{v}$ the Lie algebra algebraically generated by $\mathcal{V}$. The iterated Lie brackets of vector fields belonging to $\mathcal{V}$ linearly generate the vector space $\mathfrak{v}$. It follows from Lemma 10 that these iterated Lie brackets of vector fields are strongly tangent to the group $\mathcal{G}$. Hence $\mathfrak{v}$ is linearly generated by vector fields strongly tangent to $\mathcal{G}$. ■

Lemma 12 *Linear combinations of vector fields tangent to $\mathcal{G}$ are tangent to $\mathcal{G}$.*

Proof. If X and Y are vector fields tangent to $\mathcal{G}$ then there exist $\mathcal{C}^1$-differentiable 1-parameter families of diffeomorphisms $\{\phi_t \in \mathcal{G}\}$ and $\{\psi_t \in \mathcal{G}\}$ such that

$$\phi_0 = \psi_0 = \mathsf{Id}, \qquad \frac{\partial}{\partial t}\Big|_{t=0}\phi_t = X, \qquad \frac{\partial}{\partial t}\Big|_{t=0}\psi_t = Y.$$

Considering the $\mathcal{C}^1$-differentiable 1-parameter families of diffeomorphisms $\{\phi_t \circ \psi_t\}$ and $\{\phi_{ct}\}$ one has

$$X + Y = \frac{\partial}{\partial t}\Big|_{t=0}(\phi_t \circ \psi_t), \qquad c\,X = \frac{\partial}{\partial t}\Big|_{t=0}\phi_{(c\,t)}, \quad \text{for all} \quad c \in \mathbb{R}^n,$$

which proves the assertion. ■

Lemmas 9 – 12 prove Theorem 8. ■

3.1 Diffeomorphism group of compact manifolds

The group $\mathsf{Diff}^\infty(K)$ of diffeomorphisms of a compact manifold K is an infinite dimensional Lie group belonging to the class of Fréchet Lie groups. The Lie algebra of $\mathsf{Diff}^\infty(K)$ is the Lie algebra $\mathfrak{X}^\infty(K)$ of smooth vector fields on K endowed with the negative of the usual Lie bracket of vector fields. The Fréchet Lie group $\mathsf{Diff}^\infty(K)$ is modeled on the locally convex topological Fréchet vector space $\mathfrak{X}^\infty(K)$. A sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathfrak{X}^\infty(K)$ converges to f in the topology of $\mathfrak{X}^\infty(K)$ if and only if the functions f_j and all their derivatives converge uniformly to f, respectively to the corresponding derivatives of f. We note that the difficulty of the theory of Fréchet manifolds comes from the fact that the inverse function theorem and the existence theorems of differential equations, which are well known for Banach manifolds, are not true in this category. These problems have led to the concept of regular Fréchet Lie groups (cf. H. Omori [32] Chapter III, A. Kriegl – P. W. Michor [21] Chapter VIII). The distinguishing properties of regular Fréchet Lie groups groups can be summarized as *a)* the existence smooth exponential map from the Lie algebra of the Fréchet Lie groups to the group itself, *b)* the existence of product integrals, which produces the convergence of some approximation methods for solving differential equations (cf. Section III.5. in [32], pp. 83 –89). J. Teichmann gave a detailed discussion of these properties in [40].

If K is a compact manifold then $\mathsf{Diff}^\infty(K)$ is a F-regular infinite dimensional Lie group modeled on the vector space $\mathfrak{X}^\infty(K)$. Particularly $\mathsf{Diff}^\infty(K)$ is a strong ILB-Lie group. In this category of group one can define the exponential mapping and the group structure is locally determined by the Lie algebra by the exponential mapping. The Lie algebra of $\mathsf{Diff}^\infty(K)$ is $\mathfrak{X}^\infty(K)$ equipped with the negative of the usual Lie bracket (cf. [31, 32]).

Proposition 13 *If a Lie subalgebra $\mathfrak{g}$ of the Lie algebra $\mathfrak{X}^\infty(K)$ of smooth vector fields on a compact manifold K is tangent to a subgroup $\mathcal{G}$ of the diffeomorphism group $\mathsf{Diff}^\infty(K)$ of K, then the group generated by the exponential image $\exp(\mathfrak{g})$ of $\mathfrak{g}$ is contained in the topological closure $\overline{\mathcal{G}}$ of $\mathcal{G}$ in $\mathsf{Diff}^\infty(K)$.*

Proof. Let us denote by $\big\langle \exp(\mathfrak{g})\big\rangle$ the group generated by the exponential image of $\mathfrak{g}$. For any element $X \in \mathfrak{g}$ there exists a $\mathcal{C}^1$-differentiable 1-parameter family $\{\Phi(t) \in \mathcal{G}\}_{t\in\mathbb{R}}$ of diffeomorphisms of the manifold K such that

$$\Phi(0) = \mathsf{Id} \quad \text{and} \quad \frac{\partial \Phi(s)}{\partial s}\Big|_{s=0} = X.$$

Then, considering $\Phi(t)$ as "hair" and using the argument of Corollary 5.4. in [32], p. 85, we get that

$$\Big\{\Phi\big(\frac{t}{n}\big)^n\Big\}_{t\in\mathbb{R}} = \Big\{\Phi\big(\frac{t}{n}\big)\circ\cdots\circ\Phi\big(\frac{t}{n}\big)\Big\}_{t\in\mathbb{R}} \subset \mathcal{G}, \quad n = 1, 2\ldots$$

as a sequence of $\mathsf{Diff}^\infty(K)$ converges uniformly in all derivatives to $\exp(tX)$. It follows that we have $\{\exp(tX); t \in \mathbb{R}\} \subset \overline{\mathcal{G}}$ for any $X \in \mathfrak{g}$) and therefore $\exp(\mathfrak{g}) \subset \overline{\mathcal{G}}$. Naturally, if for the generated group $\big\langle\exp(\mathfrak{g})\big\rangle$, then the containing relation is preserved, that is $\big\langle\exp(\mathfrak{g})\big\rangle \subset \overline{\mathcal{G}}$, which proves the proposition. ∎

4 Curvature algebra

4.1 Curvature vector fields at a point

Definition 14 A vector field $\xi \in \mathfrak{X}^\infty(\mathfrak{I}M)$ on the indicatrix bundle $\mathfrak{I}M$ is a *curvature vector field* of the Finsler manifold $(M, \mathcal{F})$, if there exist vector fields $X, Y \in \mathfrak{X}^\infty(M)$ on the manifold M such that $\xi = r(X, Y)$, where for every $x \in M$ and $y \in \mathfrak{I}_xM$ we have

$$r(X, Y)(x, y) := R_{(x,y)}(X_x, Y_x). \tag{23}$$

If $x \in M$ is fixed and $X, Y \in T_xM$, then the vector field $y \to r(X,Y)(x,y)$ on $\mathfrak{I}_xM$ is a *curvature vector field at the point x*.

The Lie algebra $\mathfrak{R}(M)$ of vector fields generated by the curvature vector fields of $(M, \mathcal{F})$ is called the *curvature algebra* of the Finsler manifold $(M, \mathcal{F})$. For a fixed $x \in M$ the Lie algebra $\mathfrak{R}_x$ of vector fields generated by the curvature vector fields at x is called the *curvature algebra at the point x*.

In this section we will investigate the properties of the curvature vector fields and of the curvature algebra at a fixed point $x \in M$.
Since the Finsler metric is preserved by parallel translations, its derivatives with respect to horizontal vector fields are identically zero. Using (7) we obtain, that the derivative of the Finsler metric with respect to curvature vector fields vanishes, and hence

$$g_{(x,y)}\big(y, R_{(x,y)}(l_yX, l_yY)\big) = 0, \quad \text{for any} \quad y, X, Y \in T_xM$$

(c.f. [34], eq. (10.9)). This means that the curvature vector fields $\xi = r_x(X,Y)$ are tangent to the indicatrix. In the sequel we investigate the tangential properties of the curvature algebra to the holonomy group of the canonical connection ∇ of a Finsler manifold.

Proposition 15 *Any curvature vector field at $x \in M$ is strongly tangent to the holonomy group* $\mathsf{Hol}(x)$.

Proof. Indeed, let us consider the curvature vector field $\xi := r_x(X,Y) \in \mathfrak{X}(\mathfrak{I}_xM)$ corresponding
the directions $X, Y \in T_xM$ and let $\hat{X}, \hat{Y} \in \mathfrak{X}(M)$ be commuting vector fields i.e. $[\hat{X}, \hat{Y}] = 0$ such that $\hat{X}_x = X$, $\hat{Y}_x = Y$. By the geometric construction, the flows $\{\phi_t\}$ and $\{\psi_s\}$ of $\hat{X}$ and $\hat{Y}$ are commuting, that is $\phi_s \circ \psi_t \equiv \psi_t \circ \phi_s$. For any sufficiently small $s, t \in \mathbb{R}$ we can consider the curve $\mathcal{P}_{s,t}$ defined as follows:

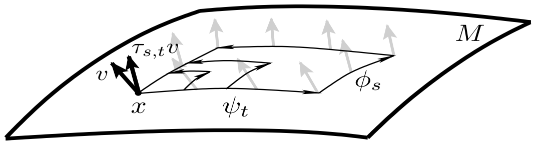


Figure 7

$$\mathcal{P}_{s,t}(u) = \begin{cases} \psi_u(x), & 0 \le u \le t, \\ \phi_u(\psi_t(x)), & t \le u \le t+s, \\ \psi_u^{-1}(\phi_s(\psi_t(x))), & t+s \le u \le 2t+s, \\ \phi_u^{-1}(\psi_t^{-1}(\phi_s(\psi_t(x)))), & 2t+s \le u \le 2t+2s. \end{cases}$$

Because of the commuting property of the flows $\{\phi_t\}$ and $\{\psi_s\}$ the curves $\mathcal{P}_{s,t}$ are closed parallelograms: their initial and final point at $u = 0$ and $u = 2t+2s$ are the same $x \in M$. Consequently, the parallel translation $\tau_{s,t} \colon T_xM \to T_xM$ along the parallelogram $\mathcal{P}_{s,t}$ is a holonomy element for every small value of $t, s \in \mathbb{R}$ (see Figure 7).

On the other hand, using the geometric construction of parallel translation presented in Section 2.1.2, we know that the flows $\{\phi_t^h\}$ and $\{\psi_s^h\}$ of the horizontal lifts $l(\hat{X})$ and $l(\hat{Y})$ can be considered as parallel translations along integral curves of $\hat{X}$ and $\hat{Y}$ respectively. They can be considered as fiber preserving diffeomorphisms of the bundle $\mathfrak{I}M$ for any $t, s \in \mathbb{R}$. Then the commutator

$$\tau_{s,t} = [\phi_s^h, \psi_t^h] = \phi_{-s}^h \circ \psi_{-t}^h \circ \phi_s^h \circ \psi_t^h : \quad \mathfrak{I}M \to \mathfrak{I}M$$

is also a fiber preserving diffeomorphism of the bundle $\mathfrak{I}M$ for any $t, s \in \mathbb{R}$. Therefore for $x \in M$ the restriction

$$\tau_{s,t}(x) = \tau_{s,t}\big|_{\mathfrak{I}_xM} : \mathfrak{I}_xM \to \mathfrak{I}_xM$$

to the fiber $\mathfrak{I}_x M$ is a 2-parameter C^∞-differentiable family of diffeomorphisms contained in the holonomy group $\mathsf{Hol}(x)$ such that

$$\tau_{s,0}(x) = \mathsf{Id}, \qquad \tau_{0,t}(x) = \mathsf{Id}, \qquad \text{and} \qquad \frac{\partial^2}{\partial t \partial s}\Big|_{s=0,t=0} \tau_{s,t}(x) = r_x(X,Y),$$

which proves that the curvature vector field $\xi = r_x(X,Y)$ is strongly tangent to the holonomy group $\mathsf{Hol}(x)$ and hence we obtain the assertion. ■

In order to enlighten the construction and the geometric meaning of the curvature vector field $\xi = r_x(X,Y)$ we consider the Figure 8 which can be seen as the extension of Figure 7. Here we can present not only the geometric objects at the level of the manifold M but at the level of the tangent manifold TM too. We remark that, as it is usual, the tangent vectors of M are represented as "arrows" at the level of M, but they are represented as "points" at the level of the tangent space TM. For example the vectors v and $\tau_{s,t}v$ are represented as arrows at x on M and points above x in TM. The gray vectors at the level of M represent the elements of the parallel vector field V along the parallelogram $\mathcal{P}_{s,t}$ with the initial condition $V_x = v$. The gray dots are the points in TM corresponding to the elements of V. These dots lie on the curves of the flows ϕ_t^h, ψ_s^h because V is a parallel field and these flows corresponds to the parallel translations along integral curves of $\hat{X}$ and $\hat{Y}$ respectively. As the picture shows, the parallel translation along the parallelogram $P_{s,t}$ of a vector $v \in T_x M$ can be obtained by following in TM the flows ϕ_t^h, ψ_s^h, ϕ_{-t}^h, ψ_{-s}^h above $P_{s,t}$. The indicatrix, or unite ball, at $x \in M$ is represented by the oval above x. Since the parallel translation preserves the norm, if $v \in \mathfrak{I}_x M$, then $\tau_{s,t}v \in \mathfrak{I}_x M$. Therefore $t \to \frac{1}{2}\tau_{t,t}(v)$ is a curve in $\mathfrak{I}_x M$. Its tangent vector at $t = 0$ is $\xi(v)$ which is therefore a tangent vector of $\mathfrak{I}_x M$.

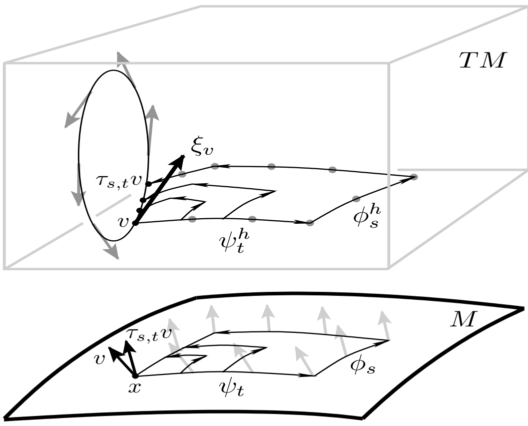


Figure 8

We remark that in the case, when the curvature is identically zero, the horizontal lifts of commuting vector fields are also commuting vector fields. Therefore one obtain $\phi_s^h \circ \psi_t^h \equiv \psi_t^h \circ \phi_s^h$ and $\tau_{s,t} = [\phi_s^h, \psi_t^h] \colon \mathfrak{I}M \to \mathfrak{I}M$ is the identity transformation. In that case $\tau_{s,t}v \equiv v$ is a constant map and therefore its derivative is zero, that is $\xi_v = 0$. Geometrically that means that the horizontal lifts of the closed parallelograms $\mathcal{P}_{s,t}$ are closed parallelograms. See Figure 9.

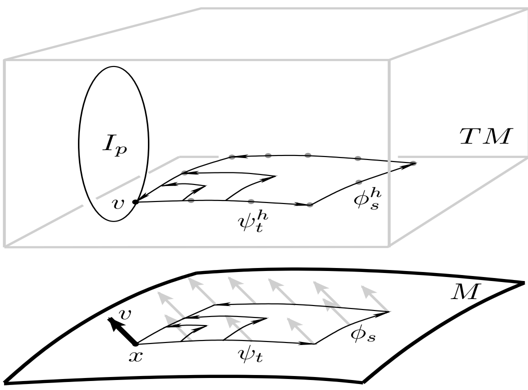


Figure 9

Theorem 16 *The curvature algebra $\mathfrak{R}_x$ at a point $x \in M$ of a Finsler manifold $(M, \mathcal{F})$ has the following properties:*

(i) $\mathfrak{R}_x$ *is tangent to the holonomy group* $\mathsf{Hol}(x)$*,*

(ii) *the group generated by the exponential image* $\exp(\mathfrak{R}_x)$ *is a subgroup of the topological closure of the holonomy group* $\mathsf{Hol}(x)$.

Proof. Since by Proposition 15 the curvature vector fields are strongly tangent to $\mathsf{Hol}(x)$ and the curvature algebra $\mathfrak{R}_x$ is algebraically generated by the curvature vector fields, the assertion (i) follows from Theorem 8. Assertion (ii) is a consequence of Proposition 13. ∎

Proposition 17 *The curvature algebra* $\mathfrak{R}_x$ *of a Riemannian manifold* (M, g) *at any point* $x \in M$ *is isomorphic to the linear Lie algebra over the vector space* T_xM *generated by the curvature operators of* (M, g) *at* $x \in M$.

Proof. The curvature tensor field of a Riemannian manifold given by the equation (7) is linear with respect to $y \in T_xM$ and hence

$$R_{(x,y)}(\xi, \eta) = (R_x(\xi, \eta))^k_l y^l \frac{\partial}{\partial y^k},$$

where $R_x(\xi, \eta))^k_l$ is the matrix of the curvature operator $R_x(\xi, \eta)\colon T_xM \to T_xM$ with respect to the natural basis $\left\{\frac{\partial}{\partial x^1}|_x, ..., \frac{\partial}{\partial x^n}|_x\right\}$. Hence any curvature vector field $r_x(\xi, \eta)(y)$ with $\xi, \eta \in T_xM$ has the shape $r_x(\xi, \eta)(y) = (R_x(\xi, \eta))^k_l y^l \frac{\partial}{\partial y^k}$. It follows that the flow of $r_x(\xi, \eta)(y)$ on the indicatrix $\mathfrak{I}_xM$ generated by the vector field $r_x(\xi, \eta)(y)$ is induced by the action of the linear 1-parameter group $\exp tR_x(\xi, \eta))$ on T_xM, which implies the assertion. ∎

Since for Finsler surfaces of non-vanishing curvature the curvature vector fields form a one-dimensional vector space and hence the generated Lie algebra is also one-dimensional, we have

Remark 18 *The curvature algebra of Finsler surfaces is at most one-dimensional.*

4.2 Constant curvature

Now, we consider a Finsler manifold $(M, \mathcal{F})$ of non-zero constant curvature. In this case for any $x \in M$ the curvature vector field $r_x(X, Y)(y)$ has the shape (cf. (14))

$$r(X, Y)(y) = c\left(\delta^i_j g_{km}(y)y^m - \delta^i_k g_{jm}(y)y^m\right) X^j Y^k \frac{\partial}{\partial y^i}, \quad 0 \neq c \in \mathbb{R}.$$

Putting $y_j = g_{jm}(y)y^m$ we can write $r(X, Y)(y) = c\left(\delta^i_j y_k - \delta^i_k y_j\right) X^j Y^k \frac{\partial}{\partial y^i}$. Any linear combination of curvature vector fields has the form

$$r(A)(y) = A^{jk}\left(\delta^i_j y_k - \delta^i_k y_j\right) \frac{\partial}{\partial y^i},$$

where $A = A^{jk} \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k} \in T_xM \wedge T_xM$ is arbitrary bivector at $x \in M$.

Lemma 19 *Let* $(M, \mathcal{F})$ *be a Finsler manifold of non-zero constant curvature. The curvature algebra* $\mathfrak{R}_x$ *at any point* $x \in M$ *satisfies*

$$\dim \mathfrak{R}_x \geq \frac{n(n-1)}{2}, \tag{24}$$

where $n = \dim M$.

Proof. Let us consider the curvature vector fields $r_{jk} = r_x(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k})(y)$ at a fixed point $x \in M$. If a linear combination

$$A^{jk} r_{jk} = A^{jk}(\delta^i_j y_k - \delta^i_k y_j)\frac{\partial}{\partial y^i} = (A^{ik} y_k - A^{ji} y_j)\frac{\partial}{\partial y^i} = 2A^{ik} y_k \frac{\partial}{\partial y^i}$$

of curvature vector fields r_{jk} with constant coefficients $A^{jk} = -A^{kj}$ satisfies $A^{jk} r_{jk} = 0$ for any $y \in T_x M$ then one has the linear equation $A^{ik} y_k = 0$ for any fixed index i. Since the covector fields $y_1, \dots, y_n$ are linearly independent we obtain $A^{jk} = 0$ for all $j, k \in \{1, \dots, n\}$. It follows that the curvature vector fields r_{jk} are linearly independent for any $j < k$ and hence $\dim \mathfrak{R}_x \geq \frac{n(n-1)}{2}$. ∎

Corollary 20 *Let (M, g) be a Riemannian manifold of non-zero constant curvature with $n = \dim M$. The curvature algebra $\mathfrak{R}_x$ at any point $x \in M$ is isomorphic to the orthogonal Lie algebra $\mathfrak{o}(n)$.*

Proof. The holonomy group of a Riemannian manifold is a subgroup of the orthogonal group $O(n)$ of the tangent space $T_x M$ and hence the curvature algebra $\mathfrak{R}_x$ is a subalgebra of the orthogonal Lie algebra $\mathfrak{o}(n)$. Hence the previous assertion implies the corollary. ∎

Theorem 21 *Let $(M, \mathcal{F})$ be a Finsler manifold of non-zero constant curvature with $n = \dim M > 2$. If the point $x \in M$ is not (semi-)Riemannian then the curvature algebra $\mathfrak{R}_x$ at $x \in M$ satisfies*

$$\dim \mathfrak{R}_x > \frac{n(n-1)}{2}. \tag{25}$$

Proof. We assume $\dim \mathfrak{R}_x = \frac{n(n-1)}{2}$. For any constant skew-symmetric matrices $\{A^{jk}\}$ and $\{B^{jk}\}$ the Lie bracket of vector fields $A^{ik} y_k \frac{\partial}{\partial y^i}$ and $B^{ik} y_k \frac{\partial}{\partial y^i}$ has the shape $C^{ik} y_k \frac{\partial}{\partial y^i}$, where $\{C^{ik}\}$ is a constant skew-symmetric matrix, too. Using the homogeneity of g_{hl} we obtain

$$\frac{\partial y_h}{\partial y^m} = \frac{\partial g_{hl}}{\partial y^m} y^l + g_{hm} = g_{hm} \tag{26}$$

and hence

$$\left[A^{mk} y_k \frac{\partial}{\partial y^m}, B^{ih} y_h \frac{\partial}{\partial y^i}\right] = \left(A^{mk} B^{ih} \frac{\partial y_h}{\partial y^m} - B^{mk} A^{ih} \frac{\partial y_h}{\partial y^m}\right) y_k \frac{\partial}{\partial y^i}$$
$$= \left(B^{ih} g_{hm} A^{mk} - A^{ih} g_{hm} B^{mk}\right) y_k \frac{\partial}{\partial y^i} = C^{ik} y_k \frac{\partial}{\partial y^i}.$$

Particularly, for the skew-symmetric matrices $E^{ij}_{ab} = \delta^i_a \delta^j_b - \delta^i_b \delta^j_a$, $a, b \in \{1, \dots, n\}$, we have

$$\left[E^{ij}_{ab} y_j \frac{\partial}{\partial y^i}, E^{kl}_{cd} y_l \frac{\partial}{\partial y^k}\right] = \left(E^{ih}_{cd} g_{hm} E^{mk}_{ab} - E^{ih}_{ab} g_{hm} E^{mk}_{cd}\right) y_k \frac{\partial}{\partial y^i} = \Lambda^{im}_{ab,cd} y_m \frac{\partial}{\partial y^i},$$

where the constants $\Lambda^{ij}_{ab,cd}$ satisfy $\Lambda^{ij}_{ab,cd} = -\Lambda^{ji}_{ab,cd} = -\Lambda^{ij}_{ba,cd} = -\Lambda^{ij}_{ab,dc} = -\Lambda^{ij}_{cd,ab}$. Putting $i = a$ and computing the trace for these indices we obtain

$$(n-2)(g_{bd} y_c - g_{bc} y_d) = \Lambda^l_{b,cd} y_l, \tag{27}$$

where $\Lambda^l_{b,cd} := \Lambda^{il}_{ib,cd}$. The right hand side is a linear form in variables $y_1, \dots, y_n$. According to the identity (27) this linear form vanishes for $y_c = y_d = 0$, hence $\Lambda^l_{b,cd} = 0$ for $l \neq c, d$. Denoting $\lambda^{(c)}_{bd} := \frac{1}{n-2} \Lambda^c_{b,cd}$ (no summation for the index c) we get the identities

$$g_{bd} y_c - g_{bc} y_d = \lambda^{(c)}_{bd} y_c - \lambda^{(d)}_{bc} y_d \quad \text{(no summation for } c \text{ and } d).$$

Putting $y_d = 0$ we obtain $g_{bd}\big|_{y_d=0} = \lambda^{(c)}_{bd}$ for any $c \neq d$. It follows $\lambda^{(c)}_{bd}$ is independent of the index c $(\neq d)$. Defining $\lambda_{bd} := \lambda^{(c)}_{bd}$ with some c $(\neq d)$ we obtain from (27) the identity

$$g_{bd} y_c - g_{bc} y_d = \lambda_{bd} y_c - \lambda_{bc} y_d \tag{28}$$

for any $b, c, d \in \{1, \dots, n\}$. We have

$$\lambda_{cd}\, y_b - \lambda_{cb}\, y_d = (g_{bd}\, y_c - g_{bc}\, y_d) - (g_{db}\, y_c - g_{dc}\, y_b) = (\lambda_{bd}\, y_c - \lambda_{bc}\, y_d) - (\lambda_{db}\, y_c - \lambda_{dc}\, y_b).$$

which implies the identity

$$\begin{aligned}(\lambda_{cd}\, y_b - \lambda_{cb}\, y_d) + (\lambda_{db}\, y_c - \lambda_{dc}\, y_b) + (\lambda_{bc}\, y_d - \lambda_{bd}\, y_c) = \\ = (\lambda_{cd} - \lambda_{dc})\, y_b + (\lambda_{db} - \lambda_{bd})\, y_c + (\lambda_{bc} - \lambda_{cb})\, y_d = 0.\end{aligned} \tag{29}$$

Since $\dim M > 2$, we can consider 3 different indices b, c, d and we obtain from the identity (29) that $\lambda_{bc} = \lambda_{cb}$ for any $b, c \in \{1, \dots, n\}$.

By derivation the identity (28) we get

$$\frac{\partial g_{bd}}{\partial y_a}\, y_c - \frac{\partial g_{bc}}{\partial y_a}\, y_d + g_{bd}\, \delta^a_c - g_{bc}\, \delta^a_d = \lambda_{bd}\, \delta^a_c - \lambda_{bc}\, \delta^a_d.$$

Using (26) we obtain

$$\begin{aligned}\frac{\partial y_a}{\partial y^q}\left(\frac{\partial g_{bd}}{\partial y_a}\, y_c - \frac{\partial g_{bc}}{\partial y_a}\, y_d\right) + g_{bd}\, g_{cq} - g_{bc}\, g_{dq} = \\ = \frac{\partial g_{bd}}{\partial y^q}\, y_c - \frac{\partial g_{bc}}{\partial y^q}\, y_d + g_{bd}\, g_{cq} - g_{bc}\, g_{dq} = \lambda_{bd}\, g_{cq} - \lambda_{bc}\, g_{dq}.\end{aligned}$$

Since

$$\left(\frac{\partial g_{bd}}{\partial y^q}\, y_c - \frac{\partial g_{bc}}{\partial y^q}\, y_d\right) y^b = 0$$

we get the identity

$$y_d\, g_{cq} - y_c\, g_{dq} = \lambda_{bd}\, y^b\, g_{cq} - \lambda_{bc}\, y^b\, g_{dq}.$$

Multiplying both sides of this identity by the inverse $\{g^{qr}\}$ of the matrix $\{g_{cq}\}$ and taking the trace with respect to the indices c, r we obtain the identity

$$(n-1)\, y_d = (n-1)\lambda_{bd}\, y^b.$$

Hence we obtain that $g_{bd}\, y^b = \lambda_{bd}\, y^b$ and hence $g_{bd} = \lambda_{bd}$, which means that the point $x \in M$ is (semi-)Riemannian. From this contradiction follows the assertion. ■

Corollary 22 *The curvature algebra $\mathfrak{R}_x$ at a point $x \in M$ of a Finsler manifold $(M, \mathcal{F})$ of non-zero constant curvature satisfies*

$$\dim \mathfrak{R}_x = \frac{n(n-1)}{2}, \quad \textit{where} \quad n = \dim M, \tag{30}$$

if and only if $n = 2$ or the point $x \in M$ is (semi-)Riemannian.

Theorem 23 *Let $(M, \mathcal{F})$ be a positive definite Finsler manifold of non-zero constant curvature with $n = \dim M > 2$. The holonomy group of $(M, \mathcal{F})$ is a compact Lie group if and only if $(M, \mathcal{F})$ is a Riemannian manifold.*

Proof. We assume that the holonomy group of a Finsler manifold $(M, \mathcal{F})$ of non-zero constant curvature with $\dim M \geq 3$ is a compact Lie transformation group on the indicatrix $\mathfrak{I}_x M$. The curvature algebra $\mathfrak{R}_x$ at a point $x \in M$ is tangent to the holonomy group $\mathsf{Hol}(x)$ and hence $\dim \mathsf{Hol}(x) \geq \dim \mathfrak{R}_x$. If there exists a not (semi-)Riemannian point $x \in M$ then $\dim \mathfrak{R}_x > \frac{n(n-1)}{2}$. The $(n-1)$-dimensional indicatrix $\mathfrak{I}_x M$ at x can be equipped with a Riemannian metric which is invariant with respect to the compact Lie transformation group $\mathsf{Hol}(x)$. Since the group of isometries of an $n-1$-dimensional Riemannian manifold is of dimension at most $\frac{n(n-1)}{2}$ (cf. Kobayashi [17], p. 46,) we obtain a contradiction, which proves the assertion. ∎

Since the holonomy group of a Landsberg manifold is a subgroup of the isometry group of the indicatrix, we obtain that any Landsberg manifold of non-zero constant curvature with dimension > 2 is Riemannian (c.f. Numata [30]).

We can summarize our results as follows:

Theorem 24 *The holonomy group of any non-Riemannian positive definite Finsler manifold of non-zero constant curvature with dimension > 2 does not occur as the holonomy group of any Riemannian manifold.*

4.3 Infinite dimensional curvature algebra

Let us consider the singular (non y-global) Finsler manifold $(H_3, \mathcal{F})$, where H_3 is the 3-dimensional Heisenberg group and $\mathcal{F}$ is a left-invariant Berwald-Moór metric (c.f. [34], Example 1.1.5, p. 8).

The group H_3 can be realized as the Lie group of matrices of the form $\begin{bmatrix} 1 & x^1 & x^2 \\ 0 & 1 & x^3 \\ 0 & 0 & 1 \end{bmatrix}$, where $x = (x^1, x^2, x^3) \in \mathbb{R}^3$ and hence the multiplication can be written as

$$(x^1, x^2, x^3) \cdot (y^1, y^2, y^3) = (x^1 + y^1, x^2 + y^2 + x^1 y^3, x^3 + y^3).$$

The vector $0 = (0, 0, 0) \in \mathbb{R}^3$ gives the unit element of H_3. The Lie algebra $\mathfrak{h}_3 = T_0 H_3$ consists of matrices of the form $\begin{bmatrix} 0 & a^1 & a^2 \\ 0 & 0 & a^3 \\ 0 & 0 & 0 \end{bmatrix}$, corresponding to the tangent vector $a = a^1 \frac{\partial}{\partial x^1} + a^2 \frac{\partial}{\partial x^2} + a^3 \frac{\partial}{\partial x^3}$ at the unit element $0 \in H_3$. A left-invariant Berwald-Moór Finsler metric $\mathcal{F}$ is induced by the (singular) Minkowski functional $\mathcal{F}_0 : \mathfrak{h}_3 \to \mathbb{R}$:

$$\mathcal{F}_0(a) := \left(a^1 a^2 a^3\right)^{\frac{2}{3}}$$

of the Lie algebra in the following way: if $y = (y^1, y^2, y^3)$ is a tangent vector at $x \in H_3$, then

$$\mathcal{F}(x, y) := \mathcal{F}_0(x^{-1} y).$$

The coordinate expression of the singular (non y-global) Finsler metric $\mathcal{F}$ is

$$\mathcal{F}(x, y) = \left(y^1 \left(y^2 - x^1 y^3\right) y^3\right)^{\frac{2}{3}}.$$

Since $\mathcal{F}$ is left-invariant, the associated geometric structures (connection, geodesics, curvature) are also left-invariant and the curvature algebras at different points are isomorphic. Using the notation

$$r_x(i, j) = r_x\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big), \quad i, j = 1, 2, 3,$$

for curvature vector fields, a direct computation yields

$$r_x(1,2) = \frac{1}{4}\left(\frac{5{y^1}^2{y^3}^2}{(x^1y^3 - y^2)^3}\frac{\partial}{\partial y^1} + \frac{y^1{y^3}^2\left(3\,x^1y^3 + y^2\right)}{(y^2 - x^1y^3)^3}\frac{\partial}{\partial y^2} + \frac{4y^1{y^3}^3}{(y^2 - x^1y^3)^3}\frac{\partial}{\partial y^3}\right),$$

$$r_x(1,3) = \frac{1}{4}\left(\frac{{y^1}^2y^3\left(6\,x^1y^3 - 11\,y^2\right)}{(x^1y^3 - y^2)^3}\frac{\partial}{\partial y^1} + \frac{4y^1{y^3}^2x^1\left(2\,x^1y^3 - 3\,y^2\right)}{(y^2 - x^1y^3)^3}\frac{\partial}{\partial y^2} + \right.$$
$$\left. + \frac{y^1{y^3}^2\left(7\,x^1y^3 - 11\,y^2\right)}{(y^2 - x^1y^3)^3}\frac{\partial}{\partial y^3}\right),$$

$$r_x(2,3) = \frac{1}{4}\left(\frac{4{y^1}^3y_3}{(x^1y^3 - y^2)^3}\frac{\partial}{\partial y^1} + \frac{{y^1}^2y^3\left(6\,x^1y^3 - y^2\right)}{(y^2 - x^1y^3)^3}\frac{\partial}{\partial y^2} + \frac{5{y^1}^2{y^3}^2}{(y^2 - x^1y^3)^3}\frac{\partial}{\partial y^3}\right).$$

The curvature vector fields $r_0(i,j),\ i,j=1,2,3$, at the unit element $0\in H_3$ generate the curvature algebra $\mathfrak{r}_0$. Let us denote $Y^{k,m} := \frac{{y^1}^k {y^3}^m}{{y^2}^{k+m-1}},\ k,m\in\mathbb{N}$, and consider the vector fields

$$A^{k,m}(a^1,a^2,a^3) = a^1 Y^{k+1,m}\frac{\partial}{\partial y^1}\Big|_0 + a^2 Y^{k,m}\frac{\partial}{\partial y^2}\Big|_0 + a^3 Y^{k,m+1}\frac{\partial}{\partial y^3}\Big|_0, \tag{31}$$

with $(a^1,a^2,a^3)\in\mathbb{R}^3$ and $k,m\in\mathbb{N}$. Then the curvature vector fields $r_0(i,j)$ at $0\in H_3$ can be written in the form

$$r_0(1,2) = \frac{1}{4}A^{1,2}(-5,1,4),\ \ r_0(1,3) = \frac{1}{4}A^{1,1}(11,0,-11),\ \ r_0(2,3) = \frac{1}{4}A^{2,1}(-4,-1,5).$$

Proposition 25 *The curvature algebra $\mathfrak{r}_x$ at any point $x\in M$ is a Lie algebra of infinite dimension.*

Proof. Since the Finsler metric is left-invariant, the curvature algebras at different points are isomorphic. Therefore it is enough to prove that the curvature algebra $\mathfrak{r}_0$ at $0\in H_3$ has infinite dimension. We prove the statement by contradiction: let us suppose that $\mathfrak{r}_0$ is finite dimensional.
A direct computation shows that for any $(a^1,a^2,a^3),(b^1,b^2,b^3)\in\mathbb{R}^3$ one has

$$\big[A^{k,m}(a^1,a^2,a^3), A^{p,q}(b^1,b^2,b^3)\big] = A^{k+p,m+q}(c^1,c^2,c^3)$$

with some $(c^1,c^2,c^3)\in\mathbb{R}^3$. It follows that any iterated Lie bracket of curvature vector fields $r_0(i,j),\ i,j=1,2,3$, has the shape (31) and hence there exists a basis of the curvature algebra $\mathfrak{r}_0$ of the form $\{A^{k_i,m_i}(a_i^1,a_i^2,a_i^3)\}_{i=1}^N$, where $N\in\mathbb{N}$ is the dimension of $\mathfrak{r}_0$. We can assume that $\{(k_i,m_i)\}_{i=1}^N$ forms an increasing sequence, i.e. $(k_1,m_1)\le(k_2,m_2)\le\dots\le(k_N,m_N)$ holds with respect to the lexicographical ordering of $\mathbb{N}\times\mathbb{N}$. We can consider the vector fields

$$\frac{4}{11}r_0(1,3) = A^{1,1}(1,0,-1),\ 4r_0(1,2) = A^{1,2}(-5,1,4),\ 4r_0(2,3) = A^{2,1}(-4,-1,5)$$

as the first three members of this sequence. Hence $1\le k_N, m_N$ and

$$\big[A^{1,1}(1,0,-1),\ A^{k_N,m_N}(a_N^1,a_N^2,a_N^3)\big] = A^{1+k_N,1+m_N}(c^1,c^2,c^3)$$

belongs to $\mathfrak{r}_0$, too, where $c^1 = (k_N-m_N-1)a_N^1 + 2a_N^2 - a_N^3$, $c^2 = (k_N-m_N)a_N^2$ and $c^3 = a_N^1 - 2a_N^2 + (k_N-m_N+1)a_N^3$. Since $k_N < 1+k_N$, $m_N < 1+m_N$ we have $c^1=c^2=c^3=0$ and hence the homogeneous linear system

$$\begin{aligned}0 &= (k_N-m_N-1)a_N^1 + 2a_N^2 - a_N^3,\\ 0 &= (k_N-m_N)a_N^2,\\ 0 &= a_N^1 - 2a_N^2 + (k_N-m_N+1)a_N^3\end{aligned}$$

has a solution $(a_N^1,a_N^2,a_N^3)\neq(0,0,0)$. It follows that $k_N=m_N$.
Similarly, computing the Lie bracket

$$0 = \big[A^{1,2}(-5,1,4), A^{k_N,k_N}(a_N^1,a_N^2,a_N^3)\big] = A^{1+k_N,2+k_N}(d^1,d^2,d^3)$$

Since $k_N < 1+k_N < 2+k_N$ we have $d^1=d^2=d^3=0$ giving the homogeneous linear system

$$\begin{aligned}0 =& (-3k_N+5)a_N^1 - 15a_N^2 + 10a_N^3,\\ 0 =& -a_N^1 + (3-3k_N)a_N^2 - 2a_N^3,\\ 0 =& -4a_N^1 + 12a_N^2 - (3k_N+8)a_N^3\end{aligned}$$

for (a_N^1,a_N^2,a_N^3). The determinant of this system vanishes only for $k_N=0$ which is a contradiction. ∎

Corollary 26 *The holonomy group of the Finsler manifold* $(H_3, \mathcal{F})$ *has an infinite dimensional tangent Lie algebra.*

We remark here, that it remains an interesting open question: *Is there a nonsingular (y-global) Finsler manifold whose curvature algebra is infinite dimensional?*

5 Holonomy algebra

5.1 Fibred holonomy group

Now, we introduce the notion of the fibred holonomy group of a Finsler manifold $(M, \mathcal{F})$ as a subgroup of the diffeomorphism group of the total manifold $\mathfrak{I}M$ of the bundle $(\mathfrak{I}M, \pi, M)$ and apply our results on tangent vector fields to an abstract subgroup of the diffeomorphism group to the study of tangent Lie algebras to the fibred holonomy group.

Definition 27 The *fibred holonomy group* $\mathsf{Hol}_{\mathsf{f}}(M)$ of $(M, \mathcal{F})$ consists of fibre preserving diffeomorphisms $\Phi \in \mathsf{Diff}^\infty(\mathfrak{I}M)$ of the indicatrix bundle $(\mathfrak{I}M, \pi, M)$ such that for any $p \in M$ the restriction $\Phi_p = \Phi|_{\mathfrak{I}_pM} \in \mathsf{Diff}^\infty(\mathfrak{I}_pM)$ belongs to the holonomy group $\mathsf{Hol}(p)$.

We note that the holonomy group $\mathsf{Hol}(p)$ and the fibred holonomy group $\mathsf{Hol}_{\mathsf{f}}(M)$ are topological subgroups of the infinite dimensional Lie groups $\mathsf{Diff}^\infty(\mathfrak{I}_pM)$ and $\mathsf{Diff}^\infty(\mathfrak{I}M)$ respectively.

The definition of strongly tangent vector fields yields

Remark 28 *A vector field* $\xi \in \mathfrak{X}^\infty(\mathfrak{I}M)$ *is strongly tangent to the fibred holonomy group* $\mathsf{Hol}_{\mathsf{f}}(M)$ *if and only if there exists a family* $\{\Phi_{(t_1,\dots,t_k)}|_{\mathfrak{I}M}\}_{t_i \in (-\varepsilon,\varepsilon)}$ *of fibre preserving diffeomorphisms of the bundle* $(\mathfrak{I}M, \pi, M)$ *such that for any indicatrix* $\mathfrak{I}_p$ *the induced family* $\{\Phi_{(t_1,\dots,t_k)}|_{\mathfrak{I}_pM}\}_{t_i \in (-\varepsilon,\varepsilon)}$ *of diffeomorphisms is contained in the holonomy group* $\mathsf{Hol}(p)$ *and* $\xi\big|_{\mathfrak{I}_pM}$ *is strongly tangent to* $\mathsf{Hol}(p)$.

Since $\pi\big(\Phi_{(t_1,\dots,t_k)}(p)\big) \equiv p$ and $\pi_*(\xi) = 0$ for every $p \in U$, we get the

Corollary 29 *Strongly tangent vector fields to the fibred holonomy group* $\mathsf{Hol}_{\mathsf{f}}(M)$ *are vertical vector fields. If* $\xi \in \mathfrak{X}^\infty(\mathfrak{I}M)$ *is strongly tangent to* $\mathsf{Hol}_{\mathsf{f}}(M)$ *then its restriction* $\xi_p := \xi\big|_{\mathfrak{I}_p}$ *to any indicatrix* $\mathfrak{I}_p$ *is strongly tangent to the holonomy group* $\mathsf{Hol}(p)$.

Now we prove that the first assertion of Theorem 16 on the tangential property to the holonomy group of curvature vector fields at a point can be extended to curvature vector fields defined on the full the indicatrix bundle.

Proposition 30 *If the Finsler manifold* $(M, \mathcal{F})$ *is diffeomorphic to* $\mathbb{R}^n$ *then any curvature vector field* $\xi \in \mathfrak{X}^\infty(\mathfrak{I}M)$ *of* $(M, \mathcal{F})$ *on the indicatrix bundle is strongly tangent to the fibred holonomy group* $\mathsf{Hol}_{\mathsf{f}}(M)$.

Proof. Since M is diffeomorphic to $\mathbb{R}^n$ we can identify the manifold M with the vector space $\mathbb{R}^n$. Let $\xi = r(X, Y) \in \mathfrak{X}^\infty(\mathfrak{I}\mathbb{R}^n)$ be a curvature vector field with $X, Y \in \mathfrak{X}^\infty(\mathbb{R}^n)$. According to Proposition 15 its restriction $\xi\big|_{\mathfrak{I}_p\mathbb{R}^n}$ to any indicatrix $\mathfrak{I}_p\mathbb{R}^n$ is strongly tangent to the holonomy groups $\mathsf{Hol}(p)$. We have to prove that there exists a family $\{\Phi_{(t_1,\dots,t_k)}|_{\mathfrak{I}\mathbb{R}^n}\}_{t_i \in (-\varepsilon,\varepsilon)}$ of fibre preserving diffeomorphisms of the indicatrix bundle $(\mathfrak{I}\mathbb{R}^n, \pi, \mathbb{R}^n)$ such that for any $p \in \mathbb{R}^n$ the family of diffeomorphisms induced on the indicatrix $\mathfrak{I}_p$ is contained in $\mathsf{Hol}(p)$ and $\xi\big|_{\mathfrak{I}_p\mathbb{R}^n}$ is strongly tangent to $\mathsf{Hol}(p)$.

For any $p \in \mathbb{R}^n$ and $-1 < s, t < 1$ let $\Pi(sX_p, tY_p)$ be the parallelogram in $\mathbb{R}^n$ determined by the vertexes $p, p + sX_p, p + sX_p + tY_p, p + tY_p \in \mathbb{R}^n$ and let $\tau_{\Pi(sX_p,tY_p)} : \mathfrak{I}_p \to \mathfrak{I}_p$ denote the (nonlinear) parallel translation of the indicatrix $\mathfrak{I}_p$ along the parallelogram $\Pi(sX_p, tY_p)$

with respect to the associated homogeneous (nonlinear) parallel translation of the Finsler manifold $(\mathbb{R}^n, \mathcal{F})$. Clearly we have $\tau_{\Pi(sX_p,tY_p)} = \mathsf{Id}_{\mathfrak{I}\mathbb{R}^n}$, if $s = 0$ or $t = 0$ and

$$\frac{\partial^2 \tau_{\Pi(sX_p,tY_p)}}{\partial s \partial t}\bigg|_{(s,t)=(0,0)} = \xi_p \quad \text{for every} \quad p \in \mathbb{R}^n.$$

Since $\Pi(sX_p, tY_p)$ is a differentiable field of parallelograms in $\mathbb{R}^n$, the maps $\tau_{\Pi(sX_p,tY_p)}$, $p \in \mathbb{R}^n$, $0 < s,t < 1$, define a 2-parameter family of fibre preserving diffeomorphisms of the indicatrix bundle $\mathfrak{I}\mathbb{R}^n$. The diffeomorphisms induced by the family $\{\tau_{\Pi(sX_p,tY_p)}\}_{s,t\in(-1,1)}$ on any indicatrix $\mathfrak{I}_p$ are contained in $\mathsf{Hol}(p)$. Hence the vector field $\xi \in \mathfrak{X}^\infty(\mathbb{R}^n)$ is strongly tangent to the fibred holonomy group $\mathsf{Hol}_{\mathsf{f}}(M)$, hence the assertion is proved. ∎

Corollary 31 *If M is diffeomorphic to $\mathbb{R}^n$ then the curvature algebra $\mathfrak{R}(M)$ of $(M, \mathcal{F})$ is tangent to the fibred holonomy group $\mathsf{Hol}_{\mathsf{f}}(M)$.*

5.2 Infinitesimal holonomy algebra

The following assertion shows that similarly to the Riemannian case, the curvature algebra can be extended to a larger tangent Lie algebra containing all horizontal Berwald covariant derivatives of the curvature vector fields.

Proposition 32 *If $\xi \in \mathfrak{X}^\infty(\mathfrak{I}M)$ is strongly tangent to the fibred holonomy group $\mathsf{Hol}_{\mathsf{f}}(M)$ of $(M, \mathcal{F})$, then its horizontal Berwald covariant derivative $\nabla_X \xi$ along any vector field $X \in \mathfrak{X}^\infty(M)$ is also strongly tangent to $\mathsf{Hol}_{\mathsf{f}}(M)$.*

Proof. Let τ be the (nonlinear) parallel translation along the flow φ of the vector field X, i.e. for every $p \in M$ and $t \in (-\varepsilon_p, \varepsilon_p)$ the map $\tau_t(p)\colon \mathfrak{I}_pM \to \mathfrak{I}_{\varphi_t(p)}M$ is the (nonlinear) parallel translation along the integral curve of X. If $\{\Phi_{(t_1,\dots,t_k)}\}_{t_i\in(-\varepsilon,\varepsilon)}$ is a $\mathcal{C}^\infty$-differentiable k-parameter family $\{\Phi_{(t_1,\dots,t_k)}\}_{t_i\in(-\varepsilon,\varepsilon)}$ of fibre preserving diffeomorphisms of the indicatrix bundle $(\mathfrak{I}M, \pi|_M, M)$ satisfying the conditions of Definition 29 then the commutator

$$[\Phi_{(t_1,\dots,t_k)}, \tau_{t_{k+1}}] := \Phi^{-1}_{(t_1,\dots,t_k)} \circ (\tau_{t_{k+1}})^{-1} \circ \Phi_{(t_1,\dots,t_k)} \circ \tau_{t_{k+1}}$$

of the group $\mathsf{Diff}^\infty(\mathfrak{I}M)$ fulfills $[\Phi_{(t_1,\dots,t_k)}, \tau_{t_{k+1}}] = \mathsf{Id}$, if some of its variables equals 0. Moreover

$$\frac{\partial^{k+1}[\Phi_{(t_1\dots t_k)}, \tau_{(t_{k+1})}]}{\partial t_1 \dots \partial t_{k+1}}\Bigg|_{(0\dots 0)} = -\big[\xi, X^h\big] \tag{32}$$

at any point of M, which shows that the vector field $\big[\xi, X^h\big]$ is strongly tangent to $\mathsf{Hol}_{\mathsf{f}}(M)$. Moreover, since the vector field ξ is vertical, we have $h[X^h, \xi] = 0$, and using $\nabla_X\xi := [X^h, \xi]$ we obtain

$$-[\xi, X^h] = [X^h, \xi] = v[X^h, \xi] = \nabla_X \xi$$

which yields the assertion. ∎

Definition 33 Let $\mathfrak{hol}^*(M)$ be the smallest Lie algebra of vector fields on the indicatrix bundle $\mathfrak{I}M$ satisfying the properties

(i) any curvature vector field ξ belongs to $\mathfrak{hol}^*(M)$,

(ii) if $\xi, \eta \in \mathfrak{hol}^*(M)$ then $[\xi, \eta] \in \mathfrak{hol}^*(M)$,

(iii) if $\xi \in \mathfrak{hol}^*(M)$ and $X \in \mathfrak{X}^\infty(M)$ then the horizontal Berwald covariant derivative $\nabla_X \xi$ also belongs to $\mathfrak{hol}^*(M)$.

The Lie algebra $\mathfrak{hol}^*(M) \subset \mathfrak{X}^\infty(\mathfrak{I}M)$ is called the *infinitesimal holonomy algebra* of the Finsler manifold $(M, \mathcal{F})$.

Remark 34 *The infinitesimal holonomy algebra* $\mathfrak{hol}^*(M)$ *is invariant with respect to the horizontal Berwald covariant derivation, i.e.*

$$\xi \in \mathfrak{hol}^*(M) \quad and \quad X \in \mathfrak{X}^\infty(M) \quad \Rightarrow \quad \nabla_X \xi \in \mathfrak{hol}^*(M). \tag{33}$$

The results of this sections yield the following

Theorem 35 *If* M *is diffeomorphic to* $\mathbb{R}^n$ *then the infinitesimal holonomy algebra* $\mathfrak{hol}^*(M)$ *is tangent to the fibred holonomy group* $\mathsf{Hol}_\mathfrak{f}(M)$.

Let $\mathfrak{hol}^*(M) \subset \mathfrak{X}^\infty(\mathfrak{I}M)$ be the infinitesimal holonomy algebra of the Finsler manifold $(M, \mathcal{F})$ and let p be a a given point in M.

Definition 36 The Lie algebra $\mathfrak{hol}^*(p) := \{ \xi_p \ ; \ \xi \in \mathfrak{hol}^*(M) \} \subset \mathfrak{X}^\infty(\mathfrak{I}_p M)$ of vector fields on the indicatrix $\mathfrak{I}_p M$ is called the *infinitesimal holonomy algebra at the point* $p \in M$.

Clearly, for any $p \in M$ the curvature algebra $\mathfrak{R}_p$ at $p \in M$ is contained in the infinitesimal holonomy algebra $\mathfrak{hol}^*(p)$ at $p \in M$.
The following assertion is a direct consequence of the definition. It shows that the infinitesimal holonomy algebra at a point p of $(M, \mathcal{F})$ can be calculated in a neighbourhood of p.

Remark 37 *Let* $(U, \mathcal{F}|_U)$ *be an open submanifold of* $(M, \mathcal{F})$ *such that* $U \subset M$ *is diffeomorphic to* $\mathbb{R}^n$ *and let* $p \in U$. *The infinitesimal holonomy algebras at* p *of the Finsler manifolds* $(M, \mathcal{F})$ *and* $(U, \mathcal{F}|_U)$ *coincide.*

Now, we can prove the following

Theorem 38 *The infinitesimal holonomy algebra* $\mathfrak{hol}^*(p)$ *at a point* $p \in M$ *has the following properties:*

(i) $\mathfrak{hol}^*(p)$ *is tangent to the holonomy group* $\mathsf{Hol}(p)$,

(ii) *the group generated by the exponential image* $\exp\big(\mathfrak{hol}^*(p)\big)$ *is a subgroup of the topological closure of the holonomy group* $\mathsf{Hol}(p)$.

Proof. Let $U \subset M$ be an open submanifold of M, diffeomorphic to $\mathbb{R}^n$ and containing $p \in M$. According to the previous remark we have $\mathfrak{hol}^*(p) := \{ \xi_p \ ; \ \xi \in \mathfrak{hol}_\mathfrak{f}(U) \}$. Since the fibred holonomy algebra $\mathfrak{hol}_\mathfrak{f}(U)$ is tangent to the fibred holonomy group $\mathsf{Hol}_\mathfrak{f}(U)$ we obtain assertion (i). Assertion (ii) is a consequence of Proposition 13. ∎

5.3 Holonomy algebra

Let $x(t)$, $0 \le t \le a$ be a smooth curve joining the points $q = x(0)$ and $p = x(a)$ in the Finsler manifold $(M, \mathcal{F})$. If $y(t) = \tau_t y(0) \in \mathfrak{I}_{x(t)} M$ is a parallel vector field along $x(t)$, $0 \le t \le a$, where $\tau_t : \mathfrak{I}_q M \to \mathfrak{I}_{x(t)} M$ denotes the homogeneous (nonlinear) parallel translation, then we have $D_{\dot{x}} y(t) := \Big(\frac{dy^i(t)}{dt} + G^i_j(x(t), y(t)) \dot{x}^j(t)\Big) \frac{\partial}{\partial x^i} = 0$. Considering a vector field ξ on the indicatrix $\mathfrak{I}_q M$, the map $\tau_{a*} \xi \circ \tau_a^{-1} : (p, y) \mapsto \tau_{a*} \xi(y(a))$ gives a vector field on the indicatrix $\mathfrak{I}_p M$. Hence we can formulate

Lemma 39 *For any vector field* $\xi \in \mathfrak{hol}^*(q) \subset \mathfrak{X}^\infty(\mathfrak{I}_q M)$ *in the infinitesimal holonomy algebra at* q *the vector field* $\tau_{a*} \xi \circ \tau_a^{-1} \in \mathfrak{X}^\infty(\mathfrak{I}_p M)$ *is tangent to the holonomy group* $\mathsf{Hol}(p)$.

Proof. Let $\{\phi_t \in \mathsf{Hol}(q)\}_{t \in (-\varepsilon, \varepsilon)}$ be a $\mathcal{C}^1$-differentiable 1-parameter family of diffeomorphisms of $\mathfrak{I}_q M$ belonging to the holonomy group $\mathsf{Hol}(q)$ and satisfying the conditions $\phi_0 = \mathsf{Id}$, $\frac{\partial \phi_t}{\partial t}\big|_{t=0} = \xi$. Since the 1-parameter family

$$\tau_a \circ \phi_t \circ \tau_a^{-1} \in \mathsf{Diff}^\infty(\mathfrak{I}_p M)\}_{t \in (-\varepsilon, \varepsilon)}$$

of diffeomorphisms consists of elements of the holonomy group $\mathsf{Hol}(p)$ and satisfies the conditions

$$\tau_a \circ \phi_0 \circ \tau_a^{-1} = \mathsf{Id}, \qquad \frac{\partial\left(\tau_a \circ \phi_t \circ \tau_a^{-1}\right)}{\partial t}\bigg|_{t=0} = \tau_{a*}\xi \circ \tau_a^{-1},$$

the assertion follows. ∎

Definition 40 A vector field $\mathbf{B}_\gamma\xi \in \mathfrak{X}^\infty(\mathfrak{I}_pM)$ on the indicatrix $\mathfrak{I}_pM$ will be called *the Berwald translate* of the vector field $\xi \in \mathfrak{X}^\infty(\mathfrak{I}_qM)$ along the curve $\gamma = x(t)$ if

$$\mathbf{B}_\gamma\xi = \tau_{a*}\xi \circ (\tau_a)^{-1}.$$

Remark 41 Let $y(t) = \tau_t y(0) \in \mathfrak{I}_{x(t)}M$ be a parallel vector field along $\gamma = x(t)$, $0 \leq t \leq a$, started at $y(0) \in \mathfrak{I}_{x(0)}M$. Then, the vertical vector field $\xi_t = \xi(x(t), y(t))$ along $(x(t), y(t))$ is the Berwald translate $\xi_t = \tau_{t*}\xi_0 \circ \tau_t{}^{-1}$ if and only if

$$\nabla_{\dot x}\xi = \left(\frac{\partial \xi^i(x,y)}{\partial x^j} - G^k_j(x,y)\frac{\partial \xi^i(x,y)}{\partial y^k} + G^i_{jk}(x,y)\xi^k(x,y)\right)\dot x^j\frac{\partial}{\partial y^i} = 0.$$

Now, lemma 39 yields the following

Corollary 42 *If $\xi \in \mathfrak{hol}^*(q)$ then its Berwald translate $\mathbf{B}_\gamma\xi \in \mathfrak{X}^\infty(\mathfrak{I}_pM)$ along any curve $\gamma = x(t)$, $0 \leq t \leq a$, joining $q = x(0)$ with $p = x(a)$ is tangent to the holonomy group $\mathsf{Hol}(p)$.*

This last statement motivates the following

Definition 43 The *holonomy algebra* $\mathfrak{hol}_p(M)$ of the Finsler manifold $(M, \mathcal{F})$ at the point $p \in M$ is defined by the smallest Lie algebra of vector fields on the indicatrix $\mathfrak{I}_pM$, containing the Berwald translates of all infinitesimal holonomy algebras along arbitrary curves $x(t)$, $0 \leq t \leq a$ joining any points $q = x(0)$ with the point $p = x(a)$.

Clearly, the holonomy algebras at different points of the Finsler manifold $(M, \mathcal{F})$ are isomorphic. Lemma 39, Corollary 42 and Proposition 13 yield the following

Theorem 44 *The holonomy algebra $\mathfrak{hol}_p(M)$ at a point $p \in M$ of a Finsler manifold $(M, \mathcal{F})$ has the following properties:*

(i) *$\mathfrak{hol}_p(M)$ is tangent to the holonomy group $\mathsf{Hol}(p)$,*

(ii) *the group generated by the exponential image $\exp\left(\mathfrak{hol}_p(M)\right)$ is a subgroup of the topological closure of the holonomy group $\mathsf{Hol}(p)$.*

5.4 Finsler surfaces with $\mathfrak{hol}^*(x) = \mathfrak{R}_x$

The relation between the infinitesimal holonomy algebra and the curvature algebra is enlightened by the following

Theorem 45 *Let $(M, \mathcal{F})$ be a Finsler surface with non-zero constant flag curvature. The infinitesimal holonomy algebra $\mathfrak{hol}^*(x)$ at a point $x \in M$ coincides with the curvature algebra $\mathfrak{R}_x$ at x if and only if the mean Berwald curvature $E_{(x,y)}$ of $(M, \mathcal{F})$ vanishes for any $y \in \mathfrak{I}_xM$.*

Proof. Let $U \subset M$ be a neighbourhood of $x \in M$ diffeomorphic to $\mathbb{R}^2$. Identifying U with $\mathbb{R}^2$ and considering a coordinate system (x_1, x_2) in $\mathbb{R}^2$ we can write

$$R^i_{jk}(x,y) = \lambda\left(\delta^i_j g_{km}(x,y)y^m - \delta^i_k g_{jm}(x,y)y^m\right), \quad \text{with} \quad \lambda \neq 0.$$

Since the curvature tensor field is skew-symmetric, $R_{(x,y)}$ acts on the one-dimensional wedge product $T_xM \wedge T_xM$. According to Lemma 2 the covariant derivative of the curvature vector field $\xi = R(X,Y) = \frac{1}{2}R(X\otimes Y - Y\otimes X) = R(X\wedge Y)$ can be written in the form

$$\nabla_Z\xi = \nabla_Z\left(r(X,Y)\right) = R\left(\nabla_Z(X\wedge Y)\right) = R(\nabla_Z X\wedge Y + X\wedge\nabla_Z Y),$$

where $X,Y,Z \in \mathfrak{X}(U)$. If $X = X^i\frac{\partial}{\partial x^i}$, $Y = Y^i\frac{\partial}{\partial x^i}$ and $Z = Z^i\frac{\partial}{\partial x^i}$ then we have $X\wedge Y = \frac{1}{2}\left(X^1Y^2 - X^2Y^1\right)\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}$ and hence we obtain

$$\nabla_Z\xi = R\left(\nabla_k\left((X^1Y^2 - Y^1X^2)\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\right)Z^k\right) = \tag{34}$$

$$= R\left(\frac{\partial(X^1Y^2 - Y^1X^2)}{\partial x^k}Z^k\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\right) + (X^1Y^2 - Y^1X^2)R\left(\nabla_k\left(\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\right)\right)Z^k,$$

where we denote the covariant derivative ∇_Z by ∇_k if $Z = \frac{\partial}{\partial x^k}$, $k = 1,2$. For given vector fields $X,Y,Z \in \mathfrak{X}^\infty(U)$ the expression $\frac{\partial(X^1Y^2 - Y^1X^2)}{\partial x^k}Z^k$ is a function on U. Hence there exists a function ψ on U such that

$$R\left(\frac{\partial(X^jY^h - Y^jX^h)}{\partial x^k}Z^k\frac{\partial}{\partial x^j}\wedge\frac{\partial}{\partial x^h}\right) = \psi\, R(X\wedge Y) = \psi\, R(X,Y),$$

and $\psi R(X,Y)$ is an element of the curvature algebra $\mathfrak{R}(U)$ of the submanifold $(U, \mathcal{F}|_U)$.

Now, we investigate the second term of the right hand side of (34).

$$\nabla_k\Big(\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\Big) = \Big(\nabla_k\frac{\partial}{\partial x^1}\Big)\wedge\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^1}\wedge\Big(\nabla_k\frac{\partial}{\partial x^2}\Big) =$$
$$= G^l_{k1}\frac{\partial}{\partial x^l}\wedge\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^1}\wedge G^m_{k2}\frac{\partial}{\partial x^m} = \left(G^1_{k1} + G^2_{k2}\right)\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}.$$

Hence

$$(X^1Y^2 - Y^1X^2)R\Big(\nabla_k\Big(\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\Big)\Big)Z^k = \left(G^1_{k1} + G^2_{k2}\right)Z^kR(X,Y) = \left(G^1_{k1} + G^2_{k2}\right)Z^k\xi$$

This expression belongs to the curvature algebra if and only if the function $G^1_{k1} + G^2_{k2}$ does not depend on the variable y, i.e. if and only if

$$E_{kh} = \frac{\partial\left(G^1_{k1} + G^2_{k2}\right)}{\partial y^h} = 0, \quad h,k = 1,2,$$

identically. ■

Remark 46 Let $\xi = R(X,Y)$ be a curvature vector field. Assume that the vector fields $X,Y \in \mathfrak{X}^\infty(M)$ have constant coordinate functions in a local coordinate system $(x^1,...,x^n)$ of the Finsler surface $(M,\mathcal{F})$. Then we have in this coordinate system

$$\nabla_Z\xi = \left(G^1_{k1} + G^2_{k2}\right)Z^k\xi.$$

A Finsler manifold $(M,\mathcal{F})$ is called *Randers manifold* if its Finsler function has the form $\mathcal{F} = \alpha + \beta$, where $\alpha = \sqrt{\alpha_{jk}(x)y^jy^k}$ is a Riemannian metric and $\beta = \beta_j(x)y^j$ is a linear form. Z. Shen constructed in [35] families of Randers surfaces depending on the real parameter ϵ, which are of constant flag curvature 1 on the unit sphere $S^2 \subset \mathbb{R}^3$ and of constant flag curvature -1 on a disk $\mathbb{D}^2 \subset \mathbb{R}^2$. These Finsler surfaces are not projectively flat and have vanishing S-curvature (c.f. [35], Theorems 1.1 and 1.2). Their Finsler function is defined by

$$\alpha = \frac{\sqrt{\epsilon^2h(v,y)^2 + h(y,y)\left(1 - \epsilon^2h(v,v)\right)}}{1 - \epsilon^2h(v,v)}, \quad \beta = \frac{\epsilon h(v,y)}{1 - \epsilon^2h(v,v)}, \tag{35}$$

where $h(v,y)$ is the standard metric of the sphere S^2, respectively $h(v,y)$ is the standard Klein metric on the unit disk $\mathbb{D}^2$ and v denotes the vector field defined by $(-x_2, x_1, 0)$ at $(x_1,x_2,x_3) \in S^2$, respectively by $(-x_2,x_1)$ at $(x_1,x_2) \in \mathbb{D}^2$.

Theorem 47 *For any Randers surface defined by (35) the infinitesimal holonomy algebra* $\mathfrak{hol}^*(x)$ *at a point* $x \in M$ *coincides with the curvature algebra* $\mathfrak{R}_x$.

Proof. According to Theorem 1.1 and 1.2 in [35], the above classes of not locally projectively flat Randers surfaces with non-zero constant flag curvature have vanishing S-curvature. Moreover, Proposition 6.1.3 in [34], p. 80, states that the mean Berwald curvature vanishes if and only if the S-curvature is a linear form on the surface. Hence the assertion follows from Corollary 45. ∎

6 Infinite dimensional infinitesimal holonomy algebra

6.1 Projective Finsler surfaces of constant curvature

A Finsler manifold $(M, \mathcal{F})$ of dimension 2 is called *Finsler surface*. In this case the indicatrix is 1-dimensional at any point $x \in M$, hence the curvature vector fields at $x \in M$ are proportional to any given non-vanishing curvature vector field. It follows that the curvature algebra $\mathfrak{R}_x(M)$ has a simple structure: it is at most 1-dimensional and commutative. Even in this case, the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ can be higher dimensional, or potentially infinite dimensional. For the investigation of such examples we use a classical result of S. Lie claiming that the dimension of a finite-dimensional Lie algebra of vector fields on a connected 1-dimensional manifold is less than 4 (cf. [1], Theorem 4.3.4). We obtain the following

Lemma 48 *If the infinitesimal holonomy algebra* $\mathfrak{hol}^*_x(M)$ *of a Finsler surface* $(M, \mathcal{F})$ *contains* 4 *simultaneously non-vanishing* $\mathbb{R}$*-linearly independent vector fields, then* $\mathfrak{hol}^*_x(M)$ *is infinite dimensional.*

Proof. If the infinitesimal holonomy algebra is finite-dimensional, then the dimension of the corresponding Lie group acting locally effectively on the 1-dimensional indicatrix would be at least 4, which is a contradiction. ∎

Let $(M, \mathcal{F})$ be a locally projectively flat Finsler surface of non-zero constant curvature, let (x^1, x^2) be a local coordinate system centered at $x \in M$, corresponding to the canonical coordinates of the Euclidean space which is projectively related to $(M, \mathcal{F})$ and let (y^1, y^2) be the induced coordinate system in the tangent plane T_xM.

In the sequel we identify the tangent plane T_xM with $\mathbb{R}^2$ with help of the coordinate system (y^1, y^2). We will use the euclidean norm $||(y^1, y^2)|| = \sqrt{(y^1)^2 + (y^2)^2}$ of $\mathbb{R}^2$ and the corresponding polar coordinate system (e^r, t), too.

Let $\varphi(y^1, y^2)$ be a positively 1-homogeneous function on $\mathbb{R}^2$ and let $r(t)$ be the 2π-periodic smooth function $r : \mathbb{R} \to \mathbb{R}$ determined by

$$\varphi(e^{r(t)}\cos t, e^{r(t)}\sin t) = 1 \quad \text{or} \quad \varphi(y^1, y^2) = e^{-r(t)}\sqrt{(y^1)^2 + (y^2)^2}, \tag{36}$$

where

$$\cos t = \frac{y^1}{\sqrt{(y^1)^2 + (y^2)^2}}, \quad \sin t = \frac{y^2}{\sqrt{(y^1)^2 + (y^2)^2}}, \quad \tan t = \frac{y^2}{y^1},$$

i.e. the level set $\{\varphi(y^1, y^2) \equiv 1\}$ of the 1-homogeneous function φ in $\mathbb{R}^2$ is given by the the parametrized curve $t \to (e^{r(t)}\cos t, e^{r(t)}\sin t)$.

Since the curvature κ of a smooth curve $t \to (e^{r(t)}\cos t, e^{r(t)}\sin t)$ in $\mathbb{R}^2$ is

$$\kappa = -\frac{e^r}{\sqrt{\dot{r}^2 + 1}}(\ddot{r} - \dot{r}^2 - 1), \tag{37}$$

the vanishing of the expression $\ddot{r} - \dot{r}^2 - 1$ means the infinitesimal linearity of the corresponding positively homogeneous function in $\mathbb{R}^2$.

Definition 49 *Let $\varphi(y^1, y^2)$ be a positively 1-homogeneous function on $\mathbb{R}^2$ and let $\kappa(t)$ be the curvature of the curve $t \to (e^{r(t)} \cos t, e^{r(t)} \sin t)$ defined by the equations (36). We say that $\varphi(y^1, y^2)$ is* strongly convex, *if $\kappa(t) \neq 0$ for all $t \in \mathbb{R}$.*

Conditions (A), (B), (C) in the following theorem imply that the projective factor $\mathcal{P}$ at $x_0 \in M$ is a non-linear function, and hence, according to Remark 3, $(M, \mathcal{F})$ is a non-Riemannian Finsler manifold.

Theorem 50 *Let $(M, \mathcal{F})$ be a projectively flat Finsler surface of non-zero constant curvature covered by a coordinate system (x^1, x^2). Assume that there exists a point $x_0 \in M$ such that one of the following conditions hold*

(A) *$\mathcal{F}$ induces a scalar product on $T_{x_0}M$ and the projective factor $\mathcal{P}$ at x_0 is a strongly convex positively 1-homogeneous function,*

(B) *$\mathcal{F}(x_0, y)$ is a strongly convex absolutely 1-homogeneous function on $T_{x_0}M$, and the projective factor $\mathcal{P}(x_0, y)$ on $T_{x_0}M$ satisfies $\mathcal{P}(x_0, y) = c\cdot\mathcal{F}(x_0, y)$ with $0 \neq c \in \mathbb{R}$,*

(C) *there is a projectively related Euclidean coordinate system of $(M, \mathcal{F})$ centered at x_0 and one has*
$$\mathcal{F}(0, y) = |y| \pm \langle a, y\rangle \quad \textit{and} \quad \mathcal{P}(0, y) = \frac{1}{2}\left(\pm|y| - \langle a, y\rangle\right). \tag{38}$$

Assume that the vector fields $U = U^i \frac{\partial}{\partial x^i}$, $V = V^i \frac{\partial}{\partial x^i} \in \mathfrak{X}^\infty(M)$ have constant coordinate functions and let $\xi = R(U, V)$ be the corresponding curvature vector field. Then the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ has an infinite dimensional subalgebra generated by the vector fields $\xi\big|_{x_0}$, $\nabla_1\xi|_{x_0}$, $\nabla_2\xi|_{x_0}$ and $\nabla_1\left(\nabla_2\xi\right)|_{x_0}$.

Proof. Since $(M, \mathcal{F})$ is of constant flag curvature, we can write
$$R^i_{jk}(x, y) = \lambda\left(\delta^i_j g_{km}(x, y)y^m - \delta^i_k g_{jm}(x, y)y^m\right), \quad \text{with} \quad \lambda = const.$$

According to Lemma 2 the horizontal Berwald covariant derivative $\nabla_W R$ of the tensor field $R = R^i_{jk}(x, y)dx^j \wedge dx^k \frac{\partial}{\partial x^i}$ vanishes and hence $\nabla_W R = 0$.
Since the curvature tensor field is skew-symmetric, $R_{(x,y)}$ acts on the one-dimensional wedge product $T_xM \wedge T_xM$. The covariant derivative $\nabla_W\xi$ of the curvature vector field $\xi = R(U, V) = \frac{1}{2}R(U \otimes V - V \otimes U) = R(U \wedge V)$ can be written in the form
$$\nabla_W\xi = R\left(\nabla_W(U \wedge V)\right) = R(\nabla_W U \wedge V + U \wedge \nabla_W V).$$
We have $U \wedge V = \frac{1}{2}\left(U^1V^2 - U^2V^1\right) \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$ and hence
$$\nabla_W\xi = (U^1V^2 - V^1U^2)W^k R\Big(\nabla_k\left(\tfrac{\partial}{\partial x^1} \wedge \tfrac{\partial}{\partial x^2}\right)\Big), \tag{39}$$
where $\nabla_k\xi := \nabla_{\frac{\partial}{\partial x^k}}\xi$. Since
$$\begin{aligned}\nabla_k\left(\tfrac{\partial}{\partial x^1} \wedge \tfrac{\partial}{\partial x^2}\right) &= \left(\nabla_k \tfrac{\partial}{\partial x^1}\right) \wedge \tfrac{\partial}{\partial x^2} + \tfrac{\partial}{\partial x^1} \wedge \left(\nabla_k \tfrac{\partial}{\partial x^2}\right) = \\ &= G^l_{k1} \tfrac{\partial}{\partial x^l} \wedge \tfrac{\partial}{\partial x^2} + \tfrac{\partial}{\partial x^1} \wedge G^m_{k2} \tfrac{\partial}{\partial x^m} = \left(G^1_{k1} + G^2_{k2}\right) \tfrac{\partial}{\partial x^1} \wedge \tfrac{\partial}{\partial x^2}\end{aligned}$$
we obtain
$$\nabla_W\xi = \left(G^1_{k1} + G^2_{k2}\right) W^k R(U, V) = \left(G^1_{k1} + G^2_{k2}\right) W^k\xi.$$
Since the geodesic coefficients are given by (17) we have
$$\nabla_W\xi = G^m_{km} W^k\xi = 3\frac{\partial\mathcal{P}}{\partial y^k} W^k\xi. \tag{40}$$

Hence

$$\nabla_Z\left(\nabla_W\xi\right)=3\nabla_Z\left(\frac{\partial\mathcal{P}}{\partial y^k}W^k\xi\right)=3\left\{\nabla_Z\left(\frac{\partial\mathcal{P}}{\partial y^k}W^k\right)\xi+\left(\frac{\partial\mathcal{P}}{\partial y^k}W^k\right)\left(\frac{\partial\mathcal{P}}{\partial y^l}Z^l\right)\right\}\xi.$$

Let W be a vector field with constant coordinate functions. Then, using (17) we get

$$\nabla_Z\left(\frac{\partial\mathcal{P}}{\partial y^k}W^k\right)=\left(\frac{\partial^2\mathcal{P}}{\partial x^j\partial y^k}-G^m_j\frac{\partial^2\mathcal{P}}{\partial y^m\partial y^k}\right)W^kZ^j=\left(\frac{\partial^2\mathcal{P}}{\partial x^j\partial y^k}-\mathcal{P}\frac{\partial^2\mathcal{P}}{\partial y^k\partial y^j}\right)W^kZ^j,$$

and hence

$$\nabla_Z\left(\nabla_W\xi\right)=3\left\{\frac{\partial^2\mathcal{P}}{\partial x^j\partial y^k}-\mathcal{P}\frac{\partial^2\mathcal{P}}{\partial y^k\partial y^j}+\frac{\partial\mathcal{P}}{\partial y^k}\frac{\partial\mathcal{P}}{\partial y^j}\right\}W^kZ^j\xi. \tag{41}$$

Let $x_0\in M$ be the point with coordinates $(0,0)$ in the local coordinate system of $(M,\mathcal{F})$ corresponding to the canonical coordinates of the projectively related Euclidean plane. According to Lemma 8.2.1 in [8], p.155, we have

$$\frac{\partial^2\mathcal{P}}{\partial x^1\partial y^2}-\mathcal{P}\frac{\partial^2\mathcal{P}}{\partial y^1\partial y^2}+\frac{\partial\mathcal{P}}{\partial y^1}\frac{\partial\mathcal{P}}{\partial y^2}=2\frac{\partial\mathcal{P}}{\partial y^1}\frac{\partial\mathcal{P}}{\partial y^2}-\frac{l}{2}\frac{\partial^2\mathcal{F}^2}{\partial y^1\partial y^2}=2\frac{\partial\mathcal{P}}{\partial y^1}\frac{\partial\mathcal{P}}{\partial y^2}-\lambda\,g_{12}.$$

Hence the vector fields $\xi\big|_{x_0}$, $\nabla_1\xi|_{x_0}$, $\nabla_2\xi|_{x_0}$ and $\nabla_1\left(\nabla_2\xi\right)|_{x_0}$ are linearly independent if and only if the functions

$$1,\qquad \frac{\partial\mathcal{P}}{\partial y^1}\Big|_{x_0},\qquad \frac{\partial\mathcal{P}}{\partial y^2}\Big|_{x_0},\qquad \left(2\frac{\partial\mathcal{P}}{\partial y^1}\frac{\partial\mathcal{P}}{\partial y^2}-\lambda\,g_{12}\right)\Big|_{x_0} \tag{42}$$

are linearly independent, where $g_{12}=g_y(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2})$ is the component of the metric tensor of $(M,\mathcal{F})$.

Lemma 51 *The functions* $\frac{\partial\mathcal{P}(0,y)}{\partial y^1}$, $\frac{\partial\mathcal{P}(0,y)}{\partial y^2}$ *and* $\mathcal{P}(0,y)\frac{\partial^2\mathcal{P}(0,y)}{\partial y^1\partial y^2}$ *can be expressed in the polar coordinate system* (e^r,t) *by*

$$\frac{\partial\mathcal{P}(0,y)}{\partial y^1}=(\cos t+\dot r\sin t)e^{-r},\qquad \frac{\partial\mathcal{P}(0,y)}{\partial y^2}=(\sin t-\dot r\cos t)e^{-r},$$

$$\mathcal{P}(0,y)\frac{\partial^2\mathcal{P}(0,y)}{\partial y^1\partial y^2}=(\dot r^2+1-\ddot r)e^{-2r}\sin t\cos t,$$

where the dot refers to differentiation with respect to the variable t.

Proof. We obtain from $\frac{\partial e^{-r}}{\partial y^1}=-e^{-r}\dot r\frac{\partial t}{\partial y^1}$ and from $-\frac{y^2}{(y^1)^2}=\frac{\partial}{\partial y^1}(\frac{y^2}{y^1})\frac{d\tan t}{dt}\frac{\partial t}{\partial y^1}=\frac{1}{\cos^2 t}\frac{\partial t}{\partial y^1}$ that $\frac{\partial e^{-r}}{\partial y^1}=e^{-r}\dot r\cos^2 t\frac{y^2}{(y^1)^2}=e^{-r}\dot r\frac{y^2}{(y^1)^2+(y^2)^2}$. Hence

$$\frac{\partial\mathcal{P}(0,y)}{\partial y^1}=\frac{\partial\big(e^{-r}\sqrt{(y^1)^2+(y^2)^2}\big)}{\partial y^1}=e^{-r}\Big(\dot r\frac{y^2}{\sqrt{(y^1)^2+(y^2)^2}}+\frac{y^1}{\sqrt{(y^1)^2+(y^2)^2}}\Big).$$

Similarly, we have $\frac{\partial e^{-r}}{\partial y^2}=-e^{-r}\dot r\cos^2 t\frac{1}{y^1}=-e^{-r}\dot r\frac{y^1}{(y^1)^2+(y^2)^2}$. Hence

$$\frac{\partial\mathcal{P}(0,y)}{\partial y^2}=\frac{\partial\big(e^{-r}\sqrt{(y^1)^2+(y^2)^2}\big)}{\partial y^2}=e^{-r}\Big(-\dot r\frac{y^1}{\sqrt{(y^1)^2+(y^2)^2}}+\frac{y^2}{\sqrt{(y^1)^2+(y^2)^2}}\Big).$$

Finally we have

$$\frac{\partial^2\mathcal{P}(0,y)}{\partial y^1\partial y^2}=\frac{\partial(\sin t-\dot r\cos t)e^{-r}}{\partial y^1}=(\ddot r-\dot r^2-1)e^{-r}\sin t\cos t\frac{1}{\sqrt{(y^1)^2+(y^2)^2}}.$$

Replacing φ by the function $\mathcal{P}(0,y)$ in the expression (36) we get the assertion. ∎

Lemma 52 *Let $r : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic smooth function such that the inequality $\ddot{r}(t) - \dot{r}^2(t) - 1 \neq 0$ holds on a dense subset of $\mathbb{R}$. Then the functions*

$$1,\ (\cos t + \dot{r}\sin t)e^{-r},\ (\sin t - \dot{r}\cos t)e^{-r},\ (\cos t + \dot{r}\sin t)(\sin t - \dot{r}\cos t)e^{-2r} \tag{43}$$

are linearly independent.

Proof. The derivative of $(\cos t + \dot{r}\sin t)e^{-r}$ and of $(\sin t - \dot{r}\cos t)e^{-r}$ are $(\ddot{r} - \dot{r}^2 - 1)e^{-r}\sin t$ and $(\ddot{r} - \dot{r}^2 - 1)e^{-r}\cos t$, respectively, hence the functions (43) do not vanish identically. Let us consider a linear combination

$$A + B(\cos t + \dot{r}\sin t)e^{-r} + C(\sin t - \dot{r}\cos t)e^{-r} + D(\cos t + \dot{r}\sin t)(\sin t - \dot{r}\cos t)e^{-2r} = 0$$

with constant coefficients A, B, C, D. We differentiate and divide by $e^{-t}(\ddot{r} - \dot{r}^2 - 1)$ and we have

$$B\sin t - C\cos t - D(\cos 2t + \dot{r}\sin 2t)e^{-r} = 0.$$

Putting $t = 0$ and $t = \pi$ we get $C = -De^{-r(0)} = De^{-r(\pi)}$. Since $e^{-r(0)},\ e^{-r(\pi)} > 0$ we get $C = D = 0$ and hence $A = B = C = D = 0$. ∎

Now, assume that condition (A) of Theorem 50 is fulfilled. According to Proposition 48 if the functions (42) are linearly independent, then the holonomy group $\mathsf{Hol}_{x_0}(M)$ is an infinite dimensional subgroup of $\mathsf{Diff}^\infty(\mathfrak{I}_{x_0}M)$. The function $\mathcal{F}(x_0, y)$ induces a scalar product on $T_{x_0}M$, consequently the component g_{12} of the metric tensor is constant on $T_{x_0}M$. Hence $\mathsf{Hol}_{x_0}(M)$ is infinite dimensional if the functions

$$1, \qquad \frac{\partial \mathcal{P}}{\partial y^1}\Big|_{x_0}, \qquad \frac{\partial \mathcal{P}}{\partial y^2}\Big|_{x_0}, \qquad \frac{\partial \mathcal{P}}{\partial y^1}\frac{\partial \mathcal{P}}{\partial y^2}\Big|_{x_0} \tag{44}$$

are linearly independent. This follows from Lemma 52 and hence the assertion of the theorem is true.

Assume that condition (B) is satisfied. We denote $\varphi(y) = \mathcal{F}(x_0, y)$. Using the expressions (42) we obtain that the vector fields $\xi\big|_{x_0}$, $\nabla_1\xi|_{x_0}$, $\nabla_2\xi|_{x_0}$ and $\nabla_1(\nabla_2\xi)|_{x_0}$ are linearly independent if and only if the functions

$$1, \qquad \frac{\partial \mathcal{P}}{\partial y^1}\Big|_{x_0} = c\frac{\partial \varphi}{\partial y^1}, \qquad \frac{\partial \mathcal{P}}{\partial y^2}\Big|_{x_0} = c\frac{\partial \varphi}{\partial y^2}$$
$$\left(2\frac{\partial \mathcal{P}}{\partial y^1}\frac{\partial \mathcal{P}}{\partial y^2} - \lambda\, g_{12}\right)\Big|_{x_0} = (2c^2 - \lambda)\frac{\partial \varphi}{\partial y^1}\frac{\partial \varphi}{\partial y^2} - \lambda\,\varphi\frac{\partial^2\varphi}{\partial y^1\partial y^2}$$

are linearly independent. According to Lemma 51 this is equivalent to the linear independence of the functions

$$1, \qquad (\cos t + \dot{r}\sin t)\,e^{-r}, \quad (\sin t - \dot{r}\cos t)\,e^{-r},$$
$$(2c^2 - \lambda)(\cos t + \dot{r}\sin t)(\sin t - \dot{r}\cos t)\,e^{-2r} - \lambda(\ddot{r} - \dot{r}^2 - 1)\,e^{-2r}\sin t\cos t.$$

If $r = const$ then these functions are $1,\ \cos t\, e^{-r},\ \sin t\, e^{-r},\ 2c^2\cos t\sin t\, e^{-2r}$, hence the assertion follows from Lemma 52. In the following we can assume that $r(t) \neq const$. Let $t_0 \in \mathbb{R}$ such that $\dot{r}(t_0) = 0$ and $\kappa(t_0) \neq 0$. We rotate the coordinate system at the angle $-t_0$ with respect to the euclidean norm $\sqrt{(y^1)^2 + (y^2)^2}$, then we get in the new polar coordinate system that $\dot{r}(0) = 0$ and $\kappa(0) \neq 0$. Consider the linear combination

$$\begin{aligned} &A + B(\cos t + \dot{r}\sin t)e^{-r} + C(\sin t - \dot{r}\cos t)\,e^{-r} + \\ &\quad + D\big((2c^2 - \lambda)(\cos t + \dot{r}\sin t)(\sin t - \dot{r}\cos t)\,e^{-2r} - \lambda(\ddot{r} - \dot{r}^2 - 1)\,e^{-2r}\sin t\cos t\big) = 0 \end{aligned} \tag{45}$$

with some constants A, B, C, D. Since the function φ is absolutely homogeneous, the function $r(t)$ is π-periodic. Putting $t + \pi$ into t, the value of

$$A + D(2c^2 - \lambda)(\cos t + \dot{r}\sin t)(\sin t - \dot{r}\cos t)\,e^{-2r} - \lambda(\ddot{r} - \dot{r}^2 - 1)\,e^{-2r}\sin t\cos t$$

does not change, but the value of

$$B(\cos t + \dot{r}\sin t)\, e^{-r} + C(\sin t - \dot{r}\cos t)\, e^{-r}$$

changes sign. Since Lemma 52 implies that $(\cos t + \dot{r}\sin t)\, e^{-r}$ and $(\sin t - \dot{r}\cos t)\, e^{-r}$ are linearly independent, we have $B = C = 0$ and (45) becomes

$$A\, e^{2r} + D\Big((2c^2 - \lambda)\big[-\dot{r}\cos 2t + \frac{1}{2}(1-\dot{r}^2)\sin 2t\big] - \frac{\lambda}{2}(\ddot{r} - \dot{r}^2 - 1)\sin 2t\Big) = 0. \tag{46}$$

Since $\dot{r}(0) = 0$ at $t = 0$, we have $A = 0$. If $D \neq 0$ then (46) gives

$$(2c^2 - \lambda)\big[-\dot{r}\cos 2t + \frac{1}{2}(1-\dot{r}^2)\sin 2t\big] - \frac{\lambda}{2}(\ddot{r} - \dot{r}^2 - 1)\sin 2t = 0.$$

By derivation and putting $t = 0$ we obtain

$$(2c^2 - \lambda)\big[-\ddot{r}(0) + 1\big] - \lambda(\ddot{r}(0) - 1) = 2c^2(1 - \ddot{r}(0)) = 0.$$

Using the relation (37) condition (B) gives $\kappa(0) = e^{r(0)}(1 - \ddot{r}(0)) \neq 0$, which is a contradiction. Hence $D = 0$ and the vector fields $\xi\big|_{x_0}$, $\nabla_1\xi|_{x_0}$, $\nabla_2\xi|_{x_0}$ and $\nabla_1(\nabla_2\xi)\,|_{x_0}$ are linearly independent. Using Proposition 48 we obtain the assertion.

Suppose now that the condition (C) holds. Hence we have

$$\frac{\partial\mathcal{F}}{\partial y^1}(0,y) = \frac{y^1}{|y|} \pm a^1, \qquad \frac{\partial\mathcal{F}}{\partial y^2}(0,y) = \frac{y^2}{|y|} \pm a^2, \qquad \frac{\partial^2\mathcal{F}}{\partial y^1\partial y^2}(0,y) = -\frac{y^1y^2}{|y|^3},$$

and

$$g_{12} = \left(\frac{y^1}{|y|} \pm a^1\right)\left(\frac{y^2}{|y|} \pm a^2\right) - \left(1 \pm \left\langle a, \frac{y}{|y|}\right\rangle\right)\frac{y^1y^2}{|y|^2}. \tag{47}$$

Similarly, we obtain from condition (C) that

$$\frac{\partial\mathcal{P}}{\partial y^1}(0,y) = \pm\frac{y^1}{|y|} - a^1, \quad \frac{\partial\mathcal{P}}{\partial y^2}(0,y) = \pm\frac{y^2}{|y|} - a^2.$$

Using the expressions (42) we get that the vector fields $\xi\big|_{x_0}$, $\nabla_1\xi|_{x_0}$, $\nabla_2\xi|_{x_0}$, $\nabla_1\left(\nabla_2\xi\right)|_{x_0}$ are linearly independent if and only if the functions

$$1, \qquad \frac{\partial\mathcal{P}}{\partial y^1}\Big|_{(0,y)} = \pm\frac{y^1}{|y|} - a^1, \qquad \frac{\partial\mathcal{P}}{\partial y^2}\Big|_{(0,y)} = \pm\frac{y^2}{|y|} - a^2$$

and

$$2\frac{\partial\mathcal{P}}{\partial y^1}\frac{\partial\mathcal{P}}{\partial y^2} - \lambda\, g_{12}\Big|_{(0,y)} = \mp\left\langle a, \frac{y}{|y|}\right\rangle\frac{y^1y^2}{|y|^2} + (1-\lambda)\frac{y^1y^2}{|y|^2} \mp (2+\lambda)\left(a_2\frac{y^1}{|y|} + a_1\frac{y^2}{|y|}\right) + (2-\lambda)a_1a_2$$

are linearly independent. Putting

$$\cos t = \frac{y^1}{|y|}, \quad \sin t = \frac{y^2}{|y|}$$

we obtain that this condition is true, since the trigonometric polynomials

$$1, \quad \cos t, \quad \sin t, \quad (1-\lambda)\cos t\sin t \mp (a_1\cos t + a_2\sin t)\cos t\sin t$$

are linearly independent. Hence $\mathsf{Hol}_{x_0}(M)$ is infinite dimensional. ∎

6.2 Projective Finsler manifolds of constant curvature

Now we will prove that the infinitesimal holonomy algebra of a totally geodesic submanifold of a Finsler manifold can be embedded into the infinitesimal holonomy algebra of the entire manifold. This result yields a lower estimate for the dimension of the holonomy group.

6.2.1 Totally geodesic and auto-parallel submanifolds

Lemma 53 *Let $\bar{M}$ be a totally geodesic submanifold in a spray manifold $(M, \mathcal{S})$. The curvature vector fields at any point of $\bar{M}$ can be extended to a curvature vector field of M.*

Proof. Assume that the manifolds $\bar{M}$ and M are k, respectively $n = k + p$ dimensional. Let $(x^1, \dots, x^k, x^{k+1}, \dots, x^n)$ be an adapted coordinate system, i. e. the submanifold $\bar{M}$ is locally given by the equations $x^{k+1} = \dots = x^n = 0$. Using the notation of the proof of Lemma 1 we get from equation (12) that $G^\sigma_\alpha = 0$ and $G^\sigma_{\alpha\,\beta} = 0$ for any $(x^1, \dots, x^k, 0, \dots, 0; y^1, \dots, y^k, 0, \dots, 0)$ we have

$$\frac{\partial G^\sigma_\alpha}{\partial x^\beta} - \frac{\partial G^\sigma_\beta}{\partial x^\alpha} + G^\tau_\alpha G^\sigma_{\beta\tau} - G^\tau_\beta G^\sigma_{\alpha\tau} + G^\gamma_\alpha G^\sigma_{\beta\gamma} - G^\gamma_\beta G^\sigma_{\alpha\gamma} = 0$$

at $(x^1, \dots, x^k, 0, \dots, 0; y^1, \dots, y^k, 0, \dots, 0)$. Hence the curvature tensors $\bar{K}$ and K, corresponding to the spray $\bar{\mathcal{S}}$, respectively to the spray $\mathcal{S}$ satisfy

$$\bar{K}(X,Y)(x,y) = K(X,Y)(x,y) \quad \text{if} \quad x \in \bar{M} \quad \text{and} \quad y, X, Y \in T_x\bar{M}.$$

It follows that for any given $X, Y \in T_x\bar{M}$ the curvature vector field $\bar{\xi}(y) = \bar{K}(X,Y)(x,y)$ at $x \in \bar{M}$ defined on $T_x\bar{M}$ can be extended to the curvature vector field $\xi(y) = K(X,Y)(x,y)$ at $x \in \bar{M}$ defined on T_xM. ∎

Theorem 54 *Let $\bar{M}$ be a totally geodesic 2-dimensional submanifold of a Finsler manifold $(M, \mathcal{F})$ such that the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\bar{M})$ of $\bar{M}$ is infinite dimensional. Then the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ of M is infinite dimensional.*

Proof. According to Lemma 1 any curvature vector field of $\bar{M}$ at $x \in \bar{M} \subset M$ defined on $\mathfrak{I}_x\bar{M}$ can be extended to a curvature vector field on the indicatrix $\mathfrak{I}_xM$. Hence the curvature algebra $\mathfrak{R}_x(\bar{M})$ of the submanifold $\bar{M}$ can be embedded into the curvature algebra $\mathfrak{R}_x(M)$ of the manifold $(M, \mathcal{F})$. Assume that $\bar{\xi}$ is a vector field belonging to the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\bar{M})$ which can be extended to the vector field ξ belonging to the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$. Any a vector field $\bar{X} \in \mathfrak{X}^\infty(\bar{M})$ can be extended to a vector field $X \in \mathfrak{X}^\infty(M)$, hence the horizontal Berwald covariant derivative along $\bar{X} \in \mathfrak{X}^\infty(\bar{M})$ of $\bar{\xi}$ can be extended to the Berwald horizontal covariant derivative along $X \in \mathfrak{X}^\infty(M)$ of the vector field ξ. It follows that the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\bar{M})$ of the submanifold $\bar{M}$ can be embedded into the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ of the Finsler manifold $(M, \mathcal{F})$. Consequently, $\mathfrak{hol}^*_x(M)$ is infinite dimensional and hence the holonomy group $\mathsf{Hol}_x(M)$ is an infinite dimensional subgroup of $\mathsf{Diff}^\infty(\mathfrak{I}_xM)$. ∎

This result can be applied to locally projectively flat Finsler manifolds, as they have for each tangent 2-plane a totally geodesic submanifold which is tangent to this 2-plane.

Corollary 55 *If a locally projectively flat Finsler manifold has a 2-dimensional totally geodesic submanifold satisfying one of the conditions of Theorem 50, then its infinitesimal holonomy algebra is infinite dimensional.*

According to equations (18) and (19) the projectively flat Randers manifolds of non-zero constant curvature satisfy condition (C) of Theorem 50. We can apply Corollary 55 to these manifolds and we get the following

Theorem 56 *The infinitesimal holonomy algebra of any projectively flat Randers manifolds of non-zero constant flag curvature is infinite dimensional.*

R. Bryant in [Br1], [Br2] introduced and studied complete Finsler metrics of positive curvature on S^2. He proved that there exists exactly a 2-parameter family of Finsler metrics on S^2 with curvature = 1 with great circles as geodesics. Z. Shen generalized a 1-parameter family of complete Bryant metrics to S^n satisfying

$$\mathcal{F}(0,y) = |y|\,\cos\alpha, \quad \mathcal{P}(0,y) = |y|\,\sin\alpha \tag{48}$$

with $|\alpha| < \frac{\pi}{2}$ in a coordinate neighbourhood centered at $0 \in \mathbb{R}^n$, (cf. Example 7.1. in [37] and Example 8.2.9 in [8]).
We investigate the holonomy groups of two families of metrics, containing the 1-parameter family of complete Bryant-Shen metrics (48). The first family in the following theorem is defined by condition (A), which is motivated by Theorem 8.2.3 in [8]. There is given the following construction:
If $\psi = \psi(y)$ is an arbitrary Minkowski norm on $\mathbb{R}^n$ and $\varphi = \varphi(y)$ is an arbitrary positively 1-homogeneous function on $\mathbb{R}^n$, then there exists a projectively flat Finsler metric $\mathcal{F}$ of constant flag curvature -1, defined on a neighbourhood of the origin, such that $\mathcal{F}$ and its projective factor $\mathcal{P}$ satisfy $\mathcal{F}(0,y) = \psi(y)$ and $\mathcal{P}(0,y) = \varphi(y)$.
Condition (B) in the next theorem is confirmed by Example 7 in [37], p. 1726, where it is proved that for an arbitrary given Minkowski norm φ and $|\vartheta| < \frac{\pi}{2}$ there exists a projectively flat Finsler function $\mathcal{F}$ of constant curvature = 1 defined on a neighbourhood of $0 \in \mathbb{R}^n$, such that

$$\mathcal{F}(0,y) = \varphi(y)\,\cos\vartheta \quad \text{and} \quad \mathcal{P}(0,y) = \varphi(y)\,\sin\vartheta.$$

Conditions (A) and (B) in Theorem 50 together with Corollary 55 yield the following

Theorem 57 *Let $(M,\mathcal{F})$ be a projectively flat Finsler manifold of non-zero constant curvature. Assume that there exists a point $x_0 \in M$ and a 2-dimensional totally geodesic submanifold $\bar{M}$ through x_0 such that one of the following conditions holds*

(A) *$\mathcal{F}$ induces a scalar product on $T_{x_0}\bar{M}$, and the projective factor $\mathcal{P}$ on $T_{x_0}\bar{M}$ is a strongly convex positively 1-homogeneous function,*

(B) *$\mathcal{F}(x_0,y)$ on $T_{x_0}\bar{M}$ is a strongly convex absolutely 1-homogeneous function on $T_{x_0}M$, and the projective factor $\mathcal{P}(x_0,y)$ on $T_{x_0}\bar{M}$ satisfies $\mathcal{P}(x_0,y) = c\cdot\mathcal{F}(x_0,y)$ with $0 \neq c \in \mathbb{R}$.*

Then the infinitesimal holonomy $\mathfrak{hol}^*_x(M)$ of M is infinite dimensional.

7 Dimension of the holonomy group

Let (M,F) be a positive definite Finsler manifold and $x \in M$ an arbitrary point in M. According to Proposition 3 of [25], the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ is tangent to the holonomy group $\mathsf{Hol}_x(M)$. Therefore the group generated by the exponential image of the infinitesimal holonomy algebra at $x \in M$ with respect to the exponential map $\exp_x : \mathfrak{X}^\infty(\mathfrak{I}_x M) \to \mathsf{Diff}^\infty(\mathfrak{I}_x M)$ is a subgroup of the closed holonomy group $\overline{\mathsf{Hol}_x(M)}$ (see Theorem 3.1 of [28]). Consequently, we have the following estimation on the dimensions:

$$\dim \mathfrak{hol}^*_x(M) \leq \dim \mathsf{Hol}_x(M). \tag{49}$$

Proposition 58 *The infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ of any locally projectively flat non-Riemannian Finsler surface $(M,\mathcal{F})$ of constant curvature $\lambda \neq 0$ is infinite dimensional.*

Proof. We use for the proof Lemma 48 and the notations introduced in the proof of Theorem 50 in the previous section on projective Finsler surfaces of constant curvature. Using the assertion on the vector fields (42) we obtain

Lemma 59 *For any fixed* $1 \le j,k \le 2$

$$y \to \xi(x,y), \quad y \to \nabla_1\xi(x,y), \quad y \to \nabla_2\xi(x,y), \quad y \to \nabla_j\left(\nabla_k\xi\right)(x,y), \tag{50}$$

considered as vector fields on $\mathfrak{I}_xM$*, are* $\mathbb{R}$*-linearly independent if and only if the*

$$1, \qquad \frac{\partial\mathcal{P}}{\partial y^1}, \qquad \frac{\partial\mathcal{P}}{\partial y^2}, \qquad \frac{\partial^2\mathcal{P}}{\partial y^j\partial y^k} - \frac{\lambda}{4}\, g_{jk} \tag{51}$$

are linearly independent functions on T_xM*.*

Since we assumed that the Finsler function $\mathcal{F}$ is non-Riemannian at the point x, i.e. $\mathcal{F}^2(x,y)$ is non-quadratic in y, the function $\mathcal{P}(x,y)$ is non-linear in y on T_xM (cf. eq. (17)). Let us choose a direction $y_0 = (y_0^1, y_0^2) \in T_xM$ with $y_0^1 \neq 0$, $y_0^2 \neq 0$ and having property that $\mathcal{P}$ is non-linear 1-homogeneous function in a conic neighbourhood U of y_0 in T_xM. By restricting U if it is necessary we can suppose that for any $y \in U$ we have $y^1 \neq 0$, $y^2 \neq 0$.
To avoid confusion between coordinate indexes and exponents, we rename the fiber coordinates of vectors belonging to U by $(u,v) = (y^1,y^2)$. Using the values of $\mathcal{P}$ on U we can define a 1-variable function $f = f(t)$ on an interval $I \subset \mathbb{R}$ by

$$f(t) := \frac{1}{v}\mathcal{P}(x_1,x_2,tv,v). \tag{52}$$

Then we can express $\mathcal{P}$ and its derivatives with f:

$$\begin{aligned} &\mathcal{P} = v\, f(u/v), && \frac{\partial\mathcal{P}}{\partial y^1} = f'(u/v), && \frac{\partial\mathcal{P}}{\partial y^2} = f(u/v) - \frac{u}{v}f'(u/v), \\ &\frac{\partial^2\mathcal{P}}{\partial y^1\partial y^1} = \frac{1}{v}f''(u/v), && \frac{\partial^2\mathcal{P}}{\partial y^1\partial y^2} = -\frac{u}{v^2}f''(u/v), && \frac{\partial^2\mathcal{P}}{\partial y^2\partial y^2} = \frac{u^2}{v^3}f''(u/v). \end{aligned} \tag{53}$$

Lemma 60 *The functions* $1, \frac{\partial\mathcal{P}}{\partial y^1}, \frac{\partial\mathcal{P}}{\partial y^2}$ *are linearly independent.*

Proof. A nontrivial relation $a + b\frac{\partial\mathcal{P}}{\partial y^1} + c\frac{\partial\mathcal{P}}{\partial y^2} = 0$ yields the differential equation $a + bf' + c(f - tf') = 0$. It is clear that both b and c cannot be zero. If $c \neq 0$ we get the differential equation

$$\frac{(a+cf)'}{a+cf} = \frac{1}{t - \frac{b}{c}}.$$

The solutions is $f(t) = t - (a+b)/c$ and therefore the corresponding $\mathcal{P}(u,v) = u - v(a+b)/c$ is linear which is a contradiction. If $c = 0$, then $b \neq 0$ and $f = -\frac{a}{b}t + K$. The corresponding $\mathcal{P}(u,v) = -\frac{a}{b}u + Kv$ is again linear which is a contradiction. ∎

Let us assume now, that the infinitesimal holonomy algebra is finite dimensional. We will show that this assumption leads to contradiction which will prove then, that the infinitesimal holonomy algebra is actually infinite dimensional.

Since $\mathfrak{I}_xM$ is 1-dimensional, according to the Lemma 48, the 4 vector fields in (50) are linearly dependent for any $j,k \in \{1,2\}$. Using Lemma 59 we get that the functions

$$1, \quad \mathcal{P}_1, \quad \mathcal{P}_2, \quad \mathcal{P}_j\mathcal{P}_k - \frac{\lambda}{4}g_{jk} \tag{54}$$

($\mathcal{P}_i = \frac{\partial\mathcal{P}}{\partial y^i}$, $\mathcal{P}_{jk} = \frac{\partial^2\mathcal{P}}{\partial y^j\partial y^k}$) are linearly dependent for any $j,k \in \{1,2\}$. From Lemma 60 we know, that the first three functions in (54) are linearly independent. Therefore by the assumption, the fourth function must be a linear combination of the first three, that is there exist constants $a_i, b_i, c_i \in \mathbb{R}$, $i = 1,2,3$, such that

$$\begin{aligned} \frac{\lambda}{4}\, g_{11} &= \mathcal{P}_1\mathcal{P}_1 + a_1 + b_1\mathcal{P}_1 + c_1\mathcal{P}_2, \\ \frac{\lambda}{4}\, g_{12} &= \mathcal{P}_1\mathcal{P}_2 + a_2 + b_2\mathcal{P}_1 + c_2\mathcal{P}_2, \\ \frac{\lambda}{4}\, g_{22} &= \mathcal{P}_2\mathcal{P}_2 + a_3 + b_3\mathcal{P}_1 + c_3\mathcal{P}_2. \end{aligned} \tag{55}$$

Using (??) we get $\partial_1 g_{21} - \partial_2 g_{11} = 0$ and $\partial_1 g_{22} - \partial_2 g_{12} = 0$ which yield

$$\begin{aligned} \mathcal{P}_2\mathcal{P}_{11} - \mathcal{P}_1\mathcal{P}_{12} + b_2\mathcal{P}_{11} + (c_2 - b_1)\mathcal{P}_{12} - c_1\mathcal{P}_{22} = 0, \\ \mathcal{P}_1\mathcal{P}_{22} - \mathcal{P}_2\mathcal{P}_{12} - b_3\mathcal{P}_{11} + (b_2 - c_3)\mathcal{P}_{12} + c_2\mathcal{P}_{22} = 0. \end{aligned} \tag{56}$$

Using the expressions (53) we obtain from (56) the equations

$$\begin{aligned} (f - \frac{u}{v}f')\frac{1}{v}f'' + f'\frac{u}{v^2}f'' + b_2\frac{1}{v}f'' - (c_2 - b_1)\frac{u}{v^2}f'' - c_1\frac{u^2}{v^3}f'' = 0, \\ f'\frac{u^2}{v^3}f'' + (f - \frac{u}{v}f')\frac{u}{v^2}f'' - b_3\frac{1}{v}f'' - (b_2 - c_3)\frac{u}{v^2}f'' + c_2\frac{u^2}{v^3}f'' = 0. \end{aligned} \tag{57}$$

Since by the non-linearity of $\mathcal{P}$ on U we have $f'' \neq 0$, equations (57) can divide by f''/v and we get

$$\begin{aligned} f + b_2 + (b_1 - c_2)\frac{u}{v} - \frac{c_1 u^2}{v^2} = 0 \\ \frac{u}{v}f - b_3 + (c_3 - b_2)\frac{u}{v} + \frac{c_2 u^2}{v^2} = 0. \end{aligned} \tag{58}$$

for any $t = u/v$ in an interval $I \subset \mathbb{R}$. The solution of this system of quadratic equations for the function f is $f(t) = -c_2\, t - b_2$ with $c_1 = b_3 = 0$, $b_1 = 2c_2$, $c_3 = 2b_2$. But this is a contradiction, since we supposed that by the non-linearity of P we have $f'' \neq 0$ on this interval. Hence the functions $1, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_j\mathcal{P}_k - \frac{\lambda}{4}g_{jk}$ can not be linearly dependent for any $j, k \in \{1, 2\}$, from which follows the assertion. ∎

Remark 61 *From Proposition 58 we get that if $(M, \mathcal{F})$ is non-Riemannian and $\lambda \neq 0$, then the holonomy group has an infinite dimensional tangent algebra.*

Indeed, according to Theorem 6.3 in [25] the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ is tangent to the holonomy group $\mathsf{Hol}_x(M)$, from which follows the assertion.

Now, we can prove our main result:

Theorem 62 *The holonomy group of a locally projectively flat simply connected Finsler manifold $(M, \mathcal{F})$ of constant curvature λ is finite dimensional if and only if $(M, \mathcal{F})$ is Riemannian or $\lambda = 0$.*

Proof. If $(M, \mathcal{F})$ is Riemannian then its holonomy group is a Lie subgroup of the orthogonal group and therefore it is a finite dimensional compact Lie group. If $(M, \mathcal{F})$ has zero curvature, then the horizontal distribution associated to the canonical connection in the tangent bundle is integrable and hence the holonomy group is trivial.
If $(M, \mathcal{F})$ is non-Riemannian having non-zero curvature λ, then for each tangent 2-plane $S \subset T_xM$ the manifold M has a totally geodesic submanifold $\widetilde{M} \subset M$ such that $T_x\widetilde{M} = S$. This $\widetilde{M}$ with the induced metric is a locally projectively flat Finsler surface of constant curvature λ. Therefore from Proposition 58 we get that $\mathfrak{hol}^*_x(\widetilde{M})$ is infinite dimensional. Moreover, according to Theorem 4.3 in [27], if a Finsler manifold $(M, \mathcal{F})$ has a totally geodesic 2-dimensional submanifold $\widetilde{M}$ such that the infinitesimal holonomy algebra of $\widetilde{M}$ is infinite dimensional, then the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ of the containing manifold is also infinite dimensional. Using (49) we get that $\mathsf{Hol}_x(M)$ cannot be finite dimensional. Hence the assertion is true. ∎

We note that there are examples of non-Riemannian type locally projectively flat Finsler manifolds with $\lambda = 0$ curvature, (cf. [22]).

Remark 63 *In the discussion before the previous theorem, the key condition for the Finsler metric tensor was not the positive definiteness but its non-degenerate property. Therefore Theorem 62 can be generalized as follows.*

A pair $(M, \mathcal{F})$ is called *semi-Finsler manifold* if in the definition of Finsler manifolds the positive definiteness of the Finsler metric tensor is replaced by the nondegenerate property. Then we have

Corollary 64 *The holonomy group of a locally projectively flat simply connected semi-Finsler manifold $(M, \mathcal{F})$ of constant curvature λ is finite dimensional if and only if $(M, \mathcal{F})$ is semi-Riemannian or $\lambda = 0$.*

8 Maximal holonomy

8.1 Holonomy group as a subgroup of the diffeomorphism group of the indicatrix

The group $\mathsf{Diff}^\infty(K)$ of diffeomorphisms of a compact manifold K is an infinite dimensional Lie group belonging to the class of Fréchet Lie groups. The Lie algebra of $\mathsf{Diff}^\infty(K)$ is the Lie algebra $\mathfrak{X}^\infty(K)$ of smooth vector fields on K endowed with the negative of the usual Lie bracket of vector fields. $\mathsf{Diff}^\infty(K)$ is modeled on the locally convex topological Fréchet vector space $\mathfrak{X}^\infty(K)$. A sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathfrak{X}^\infty(K)$ converges to f in the topology of $\mathfrak{X}^\infty(K)$ if and only if the vector fields f_j and all their derivatives converge uniformly to f, respectively to the corresponding derivatives of f. We note that the difficulty of the theory of Fréchet manifolds comes from the fact that the inverse function theorem and the existence theorems for differential equations, which are well known for Banach manifolds, are not true in this category. These problems have led to the concept of regular Fréchet Lie groups (cf. H. Omori [32] Chapter III, A. Kriegl – P. W. Michor [21] Chapter VIII). The distinguishing properties of regular Fréchet Lie groups can be summarized as the existence of smooth exponential map from the Lie algebra of the Fréchet Lie groups to the group itself, and the existence of product integrals, which produces the convergence of some approximation methods for solving differential equations (cf. Section III.5. in [32], pp. 83 –89). In particular $\mathsf{Diff}^\infty(K)$ is a topological group which is an inverse limit of Lie groups modeled on Banach spaces and hence it is a regular Fréchet Lie group (Corollary 5.4 in [32]).

Let H be a subgroup of the diffeomorphism group $\mathsf{Diff}^\infty(K)$ of a differentiable manifold K. A vector field $X \in \mathfrak{X}^\infty(K)$ is called *tangent to* $H \subset \mathsf{Diff}^\infty(K)$ if there exists a $\mathcal{C}^1$-differentiable 1-parameter family $\{\Phi(t) \in H\}_{t\in\mathbb{R}}$ of diffeomorphisms of K such that $\Phi(0) = \mathsf{Id}$ and $\frac{d\Phi(t)}{dt}\big|_{t=0} = X$. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{X}^\infty(K)$ is called *tangent to* H, if all elements of $\mathfrak{h}$ are tangent vector fields to H.

We denote by $(\mathfrak{I}M, \pi, M)$ the *indicatrix bundle* of the Finsler manifold $(M, \mathcal{F})$, the *indicatrix* $\mathfrak{I}_xM$ at $x \in M$ is the compact hypersurface $\mathfrak{I}_xM := \{y \in T_xM;\ \mathcal{F}(y) = 1\}$ in T_xM which is diffeomorphic to the sphere $\mathbb{S}^{n-1}$, if $\dim(M) = n$. The homogeneous (nonlinear) parallel translation $\tau_c : T_{c(0)}M \to T_{c(1)}M$ along a curve $c : [0,1] \to M$ preserves the value of the Finsler function, hence it induces a map $\tau_c\colon \mathfrak{I}_{c(0)}M \longrightarrow \mathfrak{I}_{c(1)}M$ between the indicatrices.

The *holonomy group* $\mathsf{Hol}_x(M)$ of the Finsler manifold $(M, \mathcal{F})$ at a point $x \in M$ is the subgroup of the group of diffeomorphisms $\mathsf{Diff}^\infty(\mathfrak{I}_xM)$ generated by homogeneous (nonlinear) parallel translations of $\mathfrak{I}_xM$ along piece-wise differentiable closed curves initiated at the point $x \in M$. The *closed holonomy group* is the topological closure $\overline{\mathsf{Hol}_x(M)}$ of the holonomy group with respect of the Fréchet topology of $\mathsf{Diff}^\infty(\mathfrak{I}_xM)$.
We remark that the diffeomorphism group $\mathsf{Diff}^\infty(\mathfrak{I}_xM)$ of the indicatrix $\mathfrak{I}_xM$ is a regular infinite dimensional Lie group modeled on the vector space $\mathfrak{X}^\infty(\mathfrak{I}_xM)$. In this category the group structure is locally determined by the Lie algebra $\mathfrak{X}^\infty(\mathfrak{I}_xM)$ of the Lie group $\mathsf{Diff}^\infty(\mathfrak{I}_xM)$ (cf. [21, 32]).

For any vector fields $X, Y \in \mathfrak{X}^\infty(M)$ on M the vector field $\xi = R(X,Y) \in \mathfrak{X}^\infty(\mathfrak{I}M)$ is called a *curvature vector field* of $(M, \mathcal{F})$ (see [24]). The Lie algebra $\mathfrak{R}(M)$ of vector fields generated by the curvature vector fields of $(M, \mathcal{F})$ is called the *curvature algebra* of $(M, \mathcal{F})$.

The restriction $\mathfrak{R}_x(M) := \{ \xi|_{\mathfrak{I}_xM} \ ; \ \xi \in \mathfrak{R}(M) \} \subset \mathfrak{X}^\infty(\mathfrak{I}_xM)$ of the curvature algebra to an indicatrix $\mathfrak{I}_xM$ is called the *curvature algebra at the point* $x \in M$.

The *infinitesimal holonomy algebra* of $(M, \mathcal{F})$ is the smallest Lie algebra $\mathfrak{hol}^*(M)$ of vector fields on the indicatrix bundle $\mathfrak{I}M$ satisfying the following properties

a) any curvature vector field ξ belongs to $\mathfrak{hol}^*(M)$,

b) if $\xi \in \mathfrak{hol}^*(M)$ and $X \in \mathfrak{X}^\infty(M)$ then the horizontal Berwald covariant derivative $\nabla_X \xi$ belongs to $\mathfrak{hol}^*(M)$.

The restriction $\mathfrak{hol}^*_x(M) := \{ \xi|_{\mathfrak{I}_xM} \ ; \ \xi \in \mathfrak{hol}^*(M) \} \subset \mathfrak{X}^\infty(\mathfrak{I}_xM)$ of the infinitesimal holonomy algebra to an indicatrix $\mathfrak{I}_xM$ is called the *infinitesimal holonomy algebra at the point* $x \in M$. One has $\mathfrak{R}(M) \subset \mathfrak{hol}^*(M)$ and $\mathfrak{R}_x(M) \subset \mathfrak{hol}^*_x(M)$ for any $x \in M$ (see [25]).

Roughly speaking, the image of the curvature tensor (the curvature vector fields) determines the curvature algebra, which generates (with the bracket operation and the covariant derivation) the infinitesimal holonomy algebra. Localising these object at a point $x \in M$ we obtain the curvature algebra and the infinitesimal holonomy algebra at $x \in M$.

The following assertion will be an important tool in the next discussion:

The infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ at any point $x \in M$ is tangent to the holonomy group $\mathsf{Hol}_x(M)$. (Theorem 6.3 in [25]).

The holonomy group and its topological closure are interesting geometrical object which reflects geometric properties of the Finsler manifold. In the characterization of the closed holonomy group we have the following corollary of Proposition 13:

Proposition 65 *The group generated by the exponential image of the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ at a point $x \in M$ with respect to the exponential map* $\exp : \mathfrak{X}^\infty(\mathfrak{I}_xM) \to \mathsf{Diff}^\infty(\mathfrak{I}_xM)$ *is a subgroup of the closed holonomy group* $\overline{\mathsf{Hol}_x(M)}$.

8.2 The group $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$ and the Fourier algebra

Let $(M, \mathcal{F})$ be a Finsler 2-manifold. In this case the indicatrix is diffeomorphic to the unit circle $\mathbb{S}^1$, at any point $x \in M$. Moreover, if there exists a non-vanishing curvature vector field at $x \in M$ then any other curvature vector field at $x \in M$ is proportional to it, which means that the curvature algebra is at most 1-dimensional. The infinitesimal holonomy algebra however, can be an infinite dimensional subalgebra of $\mathfrak{X}^\infty(\mathbb{S}^1)$, therefore the holonomy group can be an infinite dimensional subgroup of $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$, cf. [27].

Let $\mathbb{S}^1 = \mathbb{R} \mod 2\pi$ be the unit circle with the standard counterclockwise orientation. The group $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$ of orientation preserving diffeomorphisms of the $\mathbb{S}^1$ is the connected component of $\mathsf{Diff}^\infty(\mathbb{S}^1)$. The Lie algebra of $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$ is the Lie algebra $\mathfrak{X}^\infty(\mathbb{S}^1)$ – denoted also by $\mathsf{Vect}(\mathbb{S}^1)$ in the literature – can be written in the form $f(t)\frac{d}{dt}$, where f is a 2π-periodic smooth functions on the real line $\mathbb{R}$. A sequence $\{f_j \frac{d}{dt}\}_{j \in \mathbb{N}} \subset \mathsf{Vect}(\mathbb{S}^1)$ converges to $f \frac{d}{dt}$ in the Fréchet topology of $\mathsf{Vect}(\mathbb{S}^1)$ if and only if the functions f_j and all their derivatives converge uniformly to f, respectively to the corresponding derivatives of f. The Lie bracket on $\mathsf{Vect}(\mathbb{S}^1)$ is given by

$$\Big[f \frac{d}{dt}, g \frac{d}{dt}\Big] = \Big(g \frac{df}{dt} - \frac{dg}{dt} f\Big) \frac{d}{dt}.$$

The *Fourier algebra* $\mathsf{F}(\mathbb{S}^1)$ on $\mathbb{S}^1$ is the Lie subalgebra of $\mathsf{Vect}(\mathbb{S}^1)$ consisting of vector fields $f \frac{d}{dt}$ such that $f(t)$ has finite Fourier series, i.e. $f(t)$ is a Fourier polynomial. The vector fields $\big\{ \frac{d}{dt}, \cos nt \frac{d}{dt}, \sin nt \frac{d}{dt} \big\}_{n \in \mathbb{N}}$ provide a basis for $\mathsf{F}(\mathbb{S}^1)$. A direct computation shows that the vector fields

$$\frac{d}{dt}, \quad \cos t \frac{d}{dt}, \quad \sin t \frac{d}{dt}, \quad \cos 2t \frac{d}{dt}, \quad \sin 2t \frac{d}{dt} \tag{59}$$

generate the Lie algebra $\mathsf{F}(\mathbb{S}^1)$. The complexification $\mathsf{F}(\mathbb{S}^1)\otimes_{\mathbb{R}}\mathbb{C}$ of $\mathsf{F}(\mathbb{S}^1)$ is called the *Witt algebra* $\mathsf{W}(\mathbb{S}^1)$ on $\mathbb{S}^1$ having the natural basis $\big\{ie^{int}\frac{d}{dt}\big\}_{n\in\mathbb{Z}}$, with the Lie bracket $[ie^{imt}\frac{d}{dt}, ie^{int}\frac{d}{dt}] = i(m-n)e^{i(n-m)t}\frac{d}{dt}$.

Lemma 66 *The group* $\left\langle \overline{\exp\left(\mathsf{F}(\mathbb{S}^1)\right)} \right\rangle$ *generated by the topological closure of the exponential image of the Fourier algebra* $\mathsf{F}(\mathbb{S}^1)$ *is the orientation preserving diffeomorphism group* $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$.

Proof. The Fourier algebra $\mathsf{F}(\mathbb{S}^1)$ is a dense subalgebra of $\mathsf{Vect}(\mathbb{S}^1)$ with respect to the Fréchet topology, i.e. $\overline{\mathsf{F}(\mathbb{S}^1)} = \mathsf{Vect}(\mathbb{S}^1)$. This assertion follows from the fact that any r-times continuously differentiable function can be approximated uniformly by the arithmetical means of the partial sums of its Fourier series (cf. [16], 2.12 Theorem). The exponential mapping is continuous (c.f. Lemma 4.1 in [32], p. 79), hence we have

$$\exp\big(\mathsf{Vect}(\mathbb{S}^1)\big) = \exp\big(\overline{\mathsf{F}(\mathbb{S}^1)}\big) \subset \overline{\exp\big(\mathsf{F}(\mathbb{S}^1)\big)} \subset \mathsf{Diff}^\infty_+(\mathbb{S}^1) \tag{60}$$

which gives for the generated groups the relations

$$\big\langle\exp\big(\mathsf{Vect}(\mathbb{S}^1)\big)\big\rangle \subset \left\langle \overline{\exp\big(\mathsf{F}(\mathbb{S}^1)\big)} \right\rangle \subset \mathsf{Diff}^\infty_+(\mathbb{S}^1). \tag{61}$$

Moreover, the conjugation map $\mathsf{Ad} : \mathsf{Diff}^\infty_+(\mathbb{S}^1) \times \mathsf{Vect}(\mathbb{S}^1)$ satisfies the relation

$$h \exp s\xi\, h^{-1} = \exp s\mathsf{Ad}(h)\xi$$

for every $h \in \mathsf{Diff}^\infty_+(\mathbb{S}^1)$ and $\xi \in \mathsf{Vect}(\mathbb{S}^1)$. Clearly, the Lie algebra $\mathsf{Vect}(\mathbb{S}^1)$ is invariant under conjugation and hence the group $\big\langle\exp\big(\mathsf{Vect}(\mathbb{S}^1)\big)\big\rangle$ is also invariant under conjugation. Therefore $\big\langle\exp\big(\mathsf{Vect}(\mathbb{S}^1)\big)\big\rangle$ is a non-trivial normal subgroup of $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$. On the other hand $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$ is a simple group (cf. [15]) which means that its only non-trivial normal subgroup is itself. Therefore, we have $\big\langle\exp\big(\mathsf{Vect}(\mathbb{S}^1)\big)\big\rangle = \mathsf{Diff}^\infty_+(\mathbb{S}^1)$, and using (61) we get

$$\left\langle \overline{\exp(\mathsf{F}(\mathbb{S}^1))} \right\rangle = \mathsf{Diff}^\infty_+(\mathbb{S}^1).$$

■

8.3 Holonomy of the standard Funk plane and the Bryant-Shen 2-spheres

Using the results of the preceding chapter we can prove the following statement, which provides a useful tool for the investigation of the closed holonomy group of Finsler 2-manifolds.

Proposition 67 *If the infinitesimal holonomy algebra* $\mathfrak{hol}^*_x(M)$ *at a point* $x \in M$ *of a simply connected Finsler* 2*-manifold* $(M, \mathcal{F})$ *contains the Fourier algebra* $\mathsf{F}(\mathbb{S}^1)$ *on the indicatrix at* x*, then* $\overline{\mathsf{Hol}_x(M)}$ *is isomorphic to* $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$.

Proof. Since M is simply connected we have

$$\overline{\mathsf{Hol}_x(M)} \subset \mathsf{Diff}^\infty_+(\mathbb{S}^1). \tag{62}$$

On the other hand, using Proposition 13, we get

$$\exp\big(\mathsf{F}(\mathbb{S}^1)\big)\subset\overline{\mathsf{Hol}_x(M)} \;\Rightarrow\; \overline{\exp\big(\mathsf{F}(\mathbb{S}^1)\big)}\subset\overline{\mathsf{Hol}_x(M)} \;\Rightarrow\; \left\langle \overline{\exp\big(\mathsf{F}(\mathbb{S}^1)\big)} \right\rangle\subset\overline{\mathsf{Hol}_x(M)},$$

and from the last relation, using Lemma 66, we can obtain that

$$\mathsf{Diff}^\infty_+(\mathbb{S}^1) \subset \overline{\mathsf{Hol}_x(M)}. \tag{63}$$

Comparing (62) and (63) we get the assertion. ■

Using this proposition we can prove our main result:

Theorem 68 *Let $(M,\mathcal{F})$ be a simply connected projectively flat Finsler manifold of constant curvature $\lambda \neq 0$. Assume that there exists a point $x_0 \in M$ such that on $T_{x_0}M$ the induced Minkowski norm is an Euclidean norm, that is $\mathcal{F}(x_0,y)=\|y\|$, and the projective factor at x_0 satisfies $\mathcal{P}(x_0,y) = c\cdot\|y\|$ with $c \in \mathbb{R}$, $c\neq 0$. Then the closed holonomy group $\overline{\mathsf{Hol}_{x_0}(M)}$ at x_0 is isomorphic to $\mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$.*

Proof. Since $(M,\mathcal{F})$ is a locally projectively flat Finsler manifold of non-zero constant curvature, we can use an (x^1,x^2) local coordinate system centered at $x_0 \in M$, corresponding to the canonical coordinates of the Euclidean space which is projectively related to $(M,\mathcal{F})$. Let (y^1,y^2) be the induced coordinate system in the tangent plane T_xM. In the sequel we identify the tangent plane $T_{x_0}M$ with $\mathbb{R}^2$ by using the coordinate system (y^1,y^2). We will use the Euclidean norm $\|(y^1,y^2)\| = \sqrt{(y^1)^2+(y^2)^2}$ of $\mathbb{R}^2$ and the corresponding polar coordinate system (e^r,t), too.

Let us consider the curvature vector field ξ at $x_0 = 0$ defined by

$$\xi = R\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2}\right)\Big|_{x=0} = \lambda\big(\delta^i_2 g_{1m}(0,y)y^m - \delta^i_1 g_{2m}(0,y)y^m\big)\frac{\partial}{\partial x^i}$$

Since $(M,\mathcal{F})$ is of constant flag curvature, the horizontal Berwald covariant derivative $\nabla_W R$ of the tensor field R vanishes, c.f. Lemma 2. Therefore the covariant derivative of ξ can be written in the form

$$\nabla_W\xi = R\left(\nabla_k\left(\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\right)\right)W^k.$$

Since

$$\nabla_k\left(\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\right) = \left(G^1_{k1}+G^2_{k2}\right)\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}$$

we obtain $\nabla_W\xi = \left(G^1_{k1}+G^2_{k2}\right)W^k\xi$. Using (17) we can express $G^m_{km} = 3\frac{\partial P}{\partial y^k} = 3c\frac{y^k}{\|y\|}$ and hence

$$\nabla_k\xi = 3\frac{\partial P}{\partial y^k}\xi = 3c\frac{y^k}{\|y\|}\xi,$$

where we use the notation $\nabla_k = \nabla_{\frac{\partial}{\partial x^k}}$. Moreover we have

$$\nabla_j\left(\frac{\partial \mathcal{P}}{\partial y^k}\right) = \frac{\partial^2\mathcal{P}}{\partial x^j\partial y^k} - G^m_j\frac{\partial^2\mathcal{P}}{\partial y^m\partial y^k} = \frac{\partial^2\mathcal{P}}{\partial x^j\partial y^k} - \mathcal{P}\frac{\partial^2\mathcal{P}}{\partial y^k\partial y^j},$$

and hence

$$\nabla_j\left(\nabla_k\xi\right) = 3\left\{\frac{\partial^2\mathcal{P}}{\partial x^j\partial y^k} - \mathcal{P}\frac{\partial^2\mathcal{P}}{\partial y^k\partial y^j} + 3\frac{\partial\mathcal{P}}{\partial y^k}\frac{\partial\mathcal{P}}{\partial y^j}\right\}\xi.$$

According to Lemma 8.2.1, equation (8.25) in [8], p. 155, we obtain

$$\frac{\partial^2\mathcal{P}}{\partial x^j\partial y^k} = \frac{\partial\mathcal{P}}{\partial y^j}\frac{\partial\mathcal{P}}{\partial y^k} + \mathcal{P}\frac{\partial^2\mathcal{P}}{\partial y^j\partial y^k} - \frac{\lambda}{2}\frac{\partial^2\mathcal{F}^2}{\partial y^j\partial y^k}.$$

Using the assumptions on $\mathcal{F}$ and on the projective factor $\mathcal{P}$ we can get at x_0

$$\nabla_j\left(\nabla_k\xi\right) = 3\left(4c^2\frac{\partial\mathcal{F}}{\partial y^j}\frac{\partial\mathcal{F}}{\partial y^k} - \frac{\lambda}{2}\frac{\partial^2\mathcal{F}^2}{\partial y^j\partial y^k}\right)\xi$$

and hence

$$\nabla_j\left(\nabla_k\xi\right) = 3\left(4\,c^2\frac{y^jy^k}{\|y\|^2} - \lambda\,\delta^{jk}\right)\xi,$$

where $\delta^{jk}\in\{0,1\}$ such that $\delta^{jk}=1$ if and only if $j=k$.

Let us introduce polar coordinates $y^1 = r\cos t$, $y^2 = r\sin t$ in the tangent space $T_{x_0}M$. We can express the curvature vector field, its first and second covariant derivatives along the indicatrix curve $\{(\cos t, \sin t);\ 0 \leq t < 2\pi\}$ as follows:

$$\xi = \lambda \frac{d}{dt}, \quad \nabla_1 \xi = 3c\lambda \cos t \frac{d}{dt}, \quad \nabla_2 \xi = -3c\lambda \sin t \frac{d}{dt}, \quad \nabla_1(\nabla_2 \xi) = 12\, c^2 \lambda \sin 2t \frac{d}{dt},$$

$$\nabla_1(\nabla_1 \xi) = \lambda \left(12\, c^2 \cos^2 t - \lambda\right) \frac{d}{dt}, \quad \nabla_2(\nabla_2 \xi) = \lambda \left(12\, c^2 \sin^2 t - \lambda\right) \frac{d}{dt}.$$

Since $c\,\lambda \neq 0$, the vector fields

$$\frac{d}{dt}, \quad \cos t\, \frac{d}{dt}, \quad \sin t\, \frac{d}{dt}, \quad \cos t \sin t\, \frac{d}{dt}, \quad \cos^2 t\, \frac{d}{dt}, \quad \sin^2 t\, \frac{d}{dt}$$

are contained in the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_0}(M)$. It follows that the generator system

$$\left\{ \frac{d}{dt}, \quad \cos t\, \frac{d}{dt}, \quad \sin t\, \frac{d}{dt}, \quad \cos 2t\, \frac{d}{dt}, \quad \sin 2t\, \frac{d}{dt} \right\}$$

of the Fourier algebra $\mathsf{F}(\mathbb{S}^1)$ (c.f. equation (59)) is contained in the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_0}(M)$. Hence the assertion follows from Proposition 67. ■

We remark, that the standard Funk plane and the Bryant-Shen 2-spheres are connected, projectively flat Finsler manifolds of nonzero constant curvature. Moreover, in each of them, there exists a point $x_0 \in M$ and an adapted local coordinate system centered at x_0 with the following properties: the Finsler norm $\mathcal{F}(x_0, y)$ and the projective factor $\mathcal{P}(x_0, y)$ at x_0 are given by $\mathcal{F}(x_0, y) = \|y\|$ and by $\mathcal{P}(x_0, y) = c \cdot \|y\|$ with some constant $c \in \mathbb{R}$, $c \neq 0$, where $\|y\|$ is an Euclidean norm in the tangent space at x_0. Using Theorem 68 we can obtain

Theorem 69 *The closed holonomy groups of the standard Funk plane and of the Bryant-Shen 2-spheres are maximal, that is diffeomorphic to the orientation preserving diffeomorphism group of* $\mathbb{S}^1$.

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