THE MINIMAL BASE SIZE FOR A p-SOLVABLE LINEAR GROUP

ZOLTÁN HALASI AND ATTILA MARÓTI

Dedicated to the memory of Ákos Seress.

Abstract. Let V be a finite vector space over a finite field of order q and of characteristic p. Let $G \leq GL(V)$ be a p-solvable completely reducible linear group. Then there exists a base for G on V of size at most 2 unless $q \leq 4$ in which case there exists a base of size at most 3. This extends a recent result of Halasi and Podoski and generalizes a theorem of Seress. A generalization of a theorem of Pálfy and Wolf is also given.

1. Introduction

For a finite permutation group $H \leq \mathrm{Sym}(\Omega)$, a subset of the finite set Ω is called a base, if its pointwise stabilizer in H is the identity. The minimal base size of H (on Ω) is denoted by $b(H)$. Notice that $|H| \leq |\Omega|^{b(H)}$.

One of the highlights of the vast literature on base sizes of permutation groups is the celebrated paper of Á. Seress [21] in which it is proved that $b(H) \leq 4$ whenever H is a solvable primitive permutation group. Since a solvable primitive permutation group is of affine type, this result is equivalent to saying that a solvable irreducible linear subgroup G of $GL(V)$ has a base of size at most 3 (in its natural action on V) where V is a finite vector space.

There are a number of results on base sizes of linear groups. For example, D. Gluck and K. Magaard [10, Corollary 3.3] have shown that a subgroup G of $GL(V)$ with $(|G|, |V|) = 1$ admits a base of size at most 94. If in addition it is assumed that G is supersolvable or of odd order then $b(G) \leq 2$ by results of T. R. Wolf [24, Theorem A] and S. Dolfi [4, Theorem 1.3]. Later Dolfi [5, Theorem 1.1] and E. P. Vdovin [22, Theorem 1.1] generalized this result to solvable coprime linear groups. Finally, Z. Halasi and K. Podoski [12, Theorem 1.1] improved this result significantly, by proving that even the solvability assumption can be dropped, and $b(G) \leq 2$ for any coprime linear group G.

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We note that for a solvable subgroup G of $GL(V)$ acting completely reducibly on V we have $b(G) \leq 2$ if the Sylow 2-subgroups of GV are Abelian (see [6, Theorem 2]) or if $|G|$ is not divisible by 3 (see [25, Theorem 2.3]).

The following definition has been introduced by M. W. Liebeck and A. Shalev in [16]. For a linear group $G \leq GL(V)$ we say that $\{v_1, \ldots, v_k\} \subseteq V$ is a strong base for G if any element of G fixing $\langle v_i \rangle$ for every $1 \leq i \leq k$ is a scalar transformation. The minimal size of a strong base for G is denoted by $b^*(G)$. It is known that $b(G) \leq b^*(G) \leq b(G) + 1$ (see [16, Lemma 3.1]). Furthermore, also $b^*(G) \leq 2$ holds for coprime linear groups by [12, Lemma 3.3 and Theorem 1.1].

The following theorem generalizes the above-mentioned result of Seress [21] and extends that of Halasi and Podoski [12] to p-solvable groups.

Theorem 1.1. *Let V be a finite vector space over a field of order q and of characteristic p. If $G \leq GL(V)$ is a p-solvable group acting completely reducibly on V, then $b^*(G) \leq 2$ unless $q \leq 4$. Moreover if $q \leq 4$ then $b^*(G) \leq 3$.*

One of the motivations of Seress [21] was a famous result of P. P. Pálfy [19, Theorem 1] and Wolf [23, Theorem 3.1] stating that a solvable primitive permutation group of degree n has order at most $24^{-1/3} n^d$ where $d = 1 + \log_9(48 \cdot 24^{1/3}) = 3.243\ldots$, that is to say, a solvable irreducible subgroup G of $GL(V)$ has size at most $24^{-1/3} |V|^{d-1}$. See also the book by Manz and Wolf [17, Chap. 1, §3, Thm. 3.5.A]. (This bound is attained for infinitely many groups.) In the following we generalize this result to p-solvable linear groups G.

Theorem 1.2. *Let V be a finite vector space over a field of characteristic p. If $G \leq GL(V)$ is a p-solvable group acting completely reducibly on V, then $|G| \leq 24^{-1/3} |V|^{d-1}$ where d is as above.*

We note that the bounds in Theorem 1.1 are best possible for all values of q. Indeed, there are infinitely many irreducible solvable linear groups $G \leq GL(V)$ with $|G| > |V|^2$ for $q = 2$ or 3 (see [19, Theorem 1] or [23, Proposition 3.2]) and there are even infinitely many odd order completely reducible linear groups $G \leq GL(V)$ with $|G| > |V|$ for $q \geq 5$ (see [20, Theorem 3B] and the remark that follows). For $q = 4$ we note that [8] shows that there are primitive, irreducible solvable linear subgroups H of $GL(3, 4)$ with $b(H) = 3$ and thus there are infinitely many imprimitive, irreducible solvable linear groups $G = H \wr S \leq GL(3r, 4)$ with $b(G) = 3$ where S is a solvable transitive permutation group of degree r.

Theorem 1.1 has been applied in [2] to Gluck's conjecture.

2. Preliminaries

Throughout this paper let $\mathbb{F}_q$ be a finite field of characteristic p and let V be an n-dimensional vector space over $\mathbb{F}_q$. Furthermore, let $G \leq GL(V)$ be a linear group acting on V in the natural way, let $b(G)$ denote its minimal base size, and let $b^*(G)$ denote its minimal strong base size (both notions defined in Section 1).

If the vector space V is fixed, then the group of scalar transformations of V (the center of $GL(V)$) will be denoted by Z. Thus $Z \simeq \mathbb{F}_q^\times$, the multiplicative group of the base field. As $G \leq GL(V)$ is p-solvable if and only if GZ is p-solvable, we can (and we will) always assume, in the proofs of Theorems 1.1 and 1.2, that G contains Z. After choosing a basis $\{v_1, \ldots, v_n\} \subseteq V$, we will always identify the group $GL(V)$ with the group $GL(n, q)$.

Put $t(q) = 3$ for $q \leq 4$ and $t(q) = 2$ for $q \geq 5$.

Finally, if $G \leq GL(V)$ and $X \subseteq V$, then $C_G(X) = \{g \in G \mid g(x) = x \ \forall x \in X\}$ and $N_G(X) = \{g \in G \mid g(x) \in X \ \forall x \in X\}$ will denote the pointwise and setwise stabilizer of X in G, respectively.

3. Special bases in linear groups

In this section we will show that there exist bases of special kinds for certain linear groups. As a consequence (Corollary 3.3), we derive that it is sufficient to establish the required bounds in Theorem 1.1 for $b(G)$ rather than for $b^*(G)$.

Theorem 3.1. *Let V be an n-dimensional vector space over $\mathbb{F}_q$, a field of characteristic p and let $Z \leq G \leq GL(V)$ be a p-solvable linear group.*

(1) *If $n = 2$ and $q \geq 5$, then at least one of the following holds.*
   (a) *There is a basis $x, y \in V$ such that $N_G(\langle x \rangle) \subseteq N_G(\langle y \rangle)$.*
   (b) *$p = 2$ and there is a basis $x, y \in V$ such that $N_G(\langle x \rangle) = Z \times C_2$ and the involution g in $N_G(\langle x \rangle)$ satisfies $g(x) = x$ and $g(y) = y + x$.*

(2) *If $n = 3$ and $q = 3$ or 4, then at least one of the following holds.*
   (a) *There is a basis $x, y, z \in V$ such that $N_G(\langle x \rangle) \cap N_G(\langle y \rangle) \subseteq N_G(\langle z \rangle)$.*
   (b) *There is a basis $x, y, z \in V$ such that $N_G(\langle y, z \rangle) = G$.*

Proof. If $G \leq GL(V)$ leaves invariant a 1-dimensional subspace of V, then 1/(a) or 2/(a) is satisfied. If $n = 3$ and G leaves invariant a 2-dimensional subspace of V then 2/(b) is satisfied. Thus we may assume that G acts irreducibly on V.

If G acts imprimitively on V then it embeds in $C_{q-1} \wr S_n$ where the base group acts on $\langle x \rangle \oplus \langle y \rangle$ if $n = 2$ and on $\langle x \rangle \oplus \langle y \rangle \oplus \langle z \rangle$ if $n = 3$, for some vectors x, y, $z \in V$. In the first case $N_G(\langle x \rangle)$ is diagonalizable and thus 1/(a) is satisfied, while in the second case $N_G(\langle x \rangle) \cap N_G(\langle y \rangle)$ is diagonalizable and thus 2/(a) is satisfied. We may thus assume that G acts primitively (and irreducibly) on V.

Since G is p-solvable by assumption, we see that G does not contain $SL(V)$.

First consider statement (1). By considering the action of G on the set S of 1-dimensional subspaces of V, we may assume that the number of Sylow p-subgroups of G is equal to $|S| = q + 1$. For otherwise there exists $\langle x \rangle \in S$ whose stabilizer in G is a p'-group and thus Maschke's theorem gives 1/(a). For $q = p$ any subgroup of $GL(V)$ with $q + 1$ Sylow p-subgroups contains $SL(V)$, so in this case we are done. So assume that $q > p$.

Since G acts transitively on the set of Sylow p-subgroups of G and every Sylow p-subgroup stabilizes a unique subspace in S, it follows that G acts transitively on S. Moreover since $Z \leq G$ it also follows that G acts transitively on the set of non-zero vectors of V.

By Hering's theorem (see [13, Chapter XII, Remark 7.5 (a)]) we see that if q is odd (and not a prime by assumption) then q must be 9 and G has a normal subgroup isomorphic to $SL(2, 5)$ (case (5)). But then G is not 3-solvable and so we can rule out this possibility. Similarly, if q is even, then the only possibility is that $G \geq Z$ normalizes a Singer cycle $GL(1, q^2)$ (case (1)). The only such group not satisfying 1/(a) is the full semilinear group $\Gamma(1, q^2) \simeq GL(1, q^2).2$. In this case taking x to be any non-zero vector in V we have $N_G(\langle x \rangle) = Z \times C_2$ and the involution g in $N_G(\langle x \rangle)$ satisfies $g(x) = x$ and $g(y) = y + x$ for some $y \in V$.

Finally, statement (2) has been checked with [8] by using the list of all primitive permutation groups of degrees 27 and 64, respectively. $\square$

As a direct consequence we get the following.

Corollary 3.2. *Let us assume that $Z \le G \le GL(V)$ is a p-solvable linear group with $b(G) \le t(q)$.*

(1) *If $q \ge 5$, then one of the following holds.*
   (a) *There exists a base $x, y \in V$ such that $N_G(\langle x\rangle) \cap N_G(\langle x, y\rangle) \subseteq N_G(\langle y\rangle)$.*
   (b) *$p = 2$ and there exists a base $x, y \in V$ such that any non-identity element of $C_G(x) \cap N_G(\langle x, y\rangle)$ takes y to $y + x$.*

(2) *If $q \le 4$, then at least one of the following holds.*
   (a) *There exists a base $x, y, z \in V$ such that*
$$N_G(\langle x\rangle) \cap N_G(\langle y\rangle) \cap N_G(\langle x, y, z\rangle) \subseteq N_G(\langle z\rangle).$$
   (b) *There exists a base $x, y, z \in V$ such that $N_G(\langle x, y, z\rangle) \subseteq N_G(\langle y, z\rangle)$ with $x \notin \langle y, z\rangle$.*

Proof. First, 1/(a) or 2/(a) holds if $\dim(V) < t(q)$ so assume that $\dim(V) \ge t(q)$. Both parts of the corollary can be proved by choosing a subspace $U \le V$ of dimension $t(q)$ generated by a base for G and by restricting $N_G(U)$ to this subspace. Notice that the image of this restriction is also p-solvable, so Theorem 3.1 can be applied. □

Corollary 3.3. *Let V be a vector space over the field $\mathbb{F}_q$ of characteristic p. Let $Z \le G \le GL(V)$ be p-solvable with $b(G) \le t(q)$. Then $b^*(G) \le t(q)$.*

Proof. We may assume that $\dim(V) \ge t(q)$ and that $q > 2$. Let us choose a base for G of size $t(q)$ satisfying the property given in Corollary 3.2. For $q \ge 5$, if $x, y \in V$ is such a base, then $x, x + y$ is a strong base for G. Likewise, for $q = 3$ or 4, if $x, y, z \in V$ is a base satisfying (2/a) of Corollary 3.2, then $x, y, x + y + z$ is a strong base for G. Finally, in case $x, y, z \in V$ is a base for G satisfying (2/b) of Corollary 3.2, then $x, y + x, z + x$ is a strong base for G. □

4. Further reductions

Let us use induction on the dimension n of V in the proofs of Theorems 1.1 and 1.2. The case $n = 1$ is clear. Let us assume that $n > 1$ and that both Theorems 1.1 and 1.2 are true for dimensions less than n.

First we reduce the proof of both theorems for the case when $G \le GL(V)$ acts irreducibly on V. For otherwise let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ be a decomposition of V to irreducible $\mathbb{F}_q G$-modules.

By induction, there exist vectors $x_{i,1}, \ldots, x_{i,t(q)}$ in V_i for $1 \le i \le k$ with the property that $C_G(\{x_{i,1}, \ldots, x_{i,t(q)}\})$ is precisely the kernel of the action of G on V_i. Now put $x_j = \sum_{i=1}^k x_{i,j}$ for $1 \le j \le t(q)$. One can see that $C_G(\{x_1, \ldots, x_{t(q)}\}) = \cap_{i=1}^k C_G(V_i) = 1$.

For Theorem 1.2 notice that G is a subgroup of a direct product $\times_{i=1}^k H_i$ of p-solvable groups H_i acting irreducibly and faithfully on the V_i's. Hence we have
$$|G| \le \prod_{i=1}^k |H_i| \le \prod_{i=1}^k \left(24^{-1/3}|V_i|^{d-1}\right) = 24^{-k/3}|V|^{d-1}$$
by induction.

So from now on we will assume that $G \le GL(V)$ acts irreducibly on V.

For Theorem 1.1 we may also assume that $q \neq 2, 4$. Otherwise, G is solvable by the Odd Order Theorem and we can use the result of Seress [21].

For Theorem 1.2 we may assume that $|G| > |V|^2$. If $|G| \leq |V|^2$ then $|V|^2 < 24^{-1/3}|V|^{d-1}$ for $|V| \geq 79$, so we may assume that $|V| \leq 73$. If $|V|$ is a prime or $p = 2$ then G is solvable and the theorem of Pálfy [19] and Wolf [23] can be applied. Hence the cases $|V| = 5^2, 7^2, 3^2$ or 3^3 remain to be examined. But in these cases there is no non-solvable, p-solvable irreducible subgroup of $GL(V)$ (see [8]).

Now, if $b(G) \leq 2$ then $|G| \leq |V|^2$. So, once Theorem 1.1 is proved, it remains to prove Theorem 1.2 only in case $q = 3$ and $b(G) > 2$.

5. Imprimitive linear groups

In this section we show that we may assume (for the proofs of Theorems 1.1 and 1.2) that G is a primitive (irreducible) subgroup of $GL(V)$.

We first consider Theorem 1.1.

For $G \leq GL(V)$ an irreducible imprimitive linear group, let $V = V_1 \oplus \cdots \oplus V_k$ be a decomposition of V into subspaces such that G permutes these subspaces in a transitive and primitive way. This action of G defines a homomorphism from G into the symmetric group $\mathrm{Sym}(\Omega)$ for $\Omega = \{V_1, \ldots, V_k\}$ with kernel N.

The factor group $G/N \leq S_k$ is p-solvable, so it does not involve A_q for $q \geq 5$ and it does not involve A_5 for $q = 3$. By using [12, Theorem 2.3] it follows that for $q \geq 5$ there is a vector $a = (a_1, \ldots, a_k) \in \mathbb{F}_q^k$ such that $C_{G/N}(a) = 1$. (Here, G/N acts on $\mathbb{F}_q^k$ by permuting coordinates.) If $q = 3$ then again by [12, Theorem 2.3] we know that there is a 5 (and thus 9) part partition of Ω whose stabilizer in G/N is trivial. This implies that, for $q = 3$, there is a pair of vectors $a = (a_1, \ldots, a_k),\ b = (b_1, \ldots, b_k) \in \mathbb{F}_3^k$ such that $C_{G/N}(a) \cap C_{G/N}(b) = 1$.

In fact for $q \geq 8$ even we can say a bit more. For such a q let S be a subset of $\mathbb{F}_q$ of size $q/2$ with the property that for each $c \in \mathbb{F}_q$ exactly one of c and $c+1$ is contained in S. By [3, Lemma 1 (c)], there is a $4 \leq q/2$ part partition of Ω whose stabilizer in G is N, so there exists a vector $a = (a_1, \ldots, a_k) \in S^k$ such that $C_{G/N}(a) = 1$. (Actually, in our case, this already follows from [9, Theorem 1] by noting that since q is even, $p = 2$, and thus G/N is a solvable primitive permutation group.)

For each $1 \leq i \leq k$ let $H_i = N_G(V_i)$, so $N = \cap_i H_i$. By induction (on the dimension), there is a base in V_1 of size $t(q)$ for $H_1/C_{H_1}(V_1)$.

Now we can use Corollary 3.2. First let $q \geq 5$. Then there is a base $x_1, y_1 \in V_1$ for $K_1 = H_1/C_{H_1}(V_1) \leq GL(V_1)$ such that $N_{K_1}(\langle x_1 \rangle) \cap N_{K_1}(\langle x_1, y_1 \rangle) \subseteq N_{K_1}(\langle y_1 \rangle)$ or that any non-identity element of $C_{K_1}(x_1) \cap N_{K_1}(\langle x_1, y_1 \rangle)$ takes y_1 to $y_1 + x_1$.

Let $\{g_1 = 1, g_2, \ldots, g_k\}$ be a set of left coset representatives for H_1 in G and $x_i = g_i x_1,\ y_i = g_i y_1$ for every i. Now let

$$x = \sum_{i=1}^{k} x_i, \qquad y = \sum_{i=1}^{k} y_i + a_i x_i.$$

In case $q = 3$ let $x_1, y_1, z_1 \in V_1$ be a base for $K_1 = H_1/C_{H_1}(V_1) \leq GL(V_1)$ satisfying (2/a) or (2/b) of Corollary 3.2. Again, let $\{g_1 = 1, g_2, \ldots, g_k\}$ be a set of left coset representatives for H_1 in G and $x_i = g_i x_1,\ y_i = g_i y_1,\ z_i = g_i z_1$ for every

i. Depending on which part of part (2) of Corollary 3.2 is satisfied for x_1, y_1, z_1 let

$$x = \sum_{i=1}^{k} x_i, \qquad y = \sum_{i=1}^{k} y_i \qquad z = \sum_{i=1}^{k} (z_i + b_i x_i + a_i y_i) \quad \text{if (2/a) holds,}$$

$$x = \sum_{i=1}^{k} x_i, \qquad y = \sum_{i=1}^{k} (y_i + a_i x_i) \qquad z = \sum_{i=1}^{k} (z_i + b_i x_i) \quad \text{if (2/b) holds.}$$

In each case, it is easy to see that the given set of vectors is a base for G by using similar arguments as in the proof of [12, Theorem 2.6]. For the convenience of the reader, we present a proof here for the case (2/a).

Let x, y, z given as above and $g \in C_G(x) \cap C_G(y) \cap C_G(z)$. Furthermore, let $\sigma \in S_k$ be the permutation associated to the action of g on $\Omega = \{V_1, \dots, V_k\}$. Then $g(x) = x,\ g(y) = y$ implies that $g(x_i) = x_{\sigma(i)},\ g(y_i) = y_{\sigma(i)}$ for every $1 \le i \le k$. Using also that $g(z) = z$ we get that

$$z_{\sigma(i)} + b_{\sigma(i)} x_{\sigma(i)} + a_{\sigma(i)} y_{\sigma(i)} = g(z_i + b_i x_i + a_i y_i) = g(z_i) + b_i x_{\sigma(i)} + a_i y_{\sigma(i)},$$

thus, $g(z_i) = z_{\sigma(i)} + (b_{\sigma(i)} - b_i) x_{\sigma(i)} + (a_{\sigma(i)} - a_i) y_{\sigma(i)}$. Now, $h = g_{\sigma(i)}^{-1} g g_i \in H_1$ satisfies

$$h(x_1) = x_1,\ h(y_1) = y_1,\ h(z_1) = z_1 + (b_{\sigma(i)} - b_i) x_1 + (a_{\sigma(i)} - a_i) y_1.$$

By part (2/a) of Corollary 3.2 we conclude that $b_{\sigma(i)} = b_i$ and $a_{\sigma(i)} = a_i$ for every $1 \le i \le k$. In other words, σ fixes both $(a_1, \dots, a_k)$ and $(b_1, \dots, b_k)$. By the defintion of these vectors we get that $\sigma = 1$, i.e. $g \in \cap_i H_i = N$. Furthermore, for every $1 \le i \le k$ we also have

$$g(x_i) = x_i,\ g(y_i) = y_i,\ g(z_i + b_i x_i + a_i y_i) = z_i + b_i x_i + a_i y_i.$$

Since $x_i, y_i, z_i + b_i x_i + a_i y_i \in V_i$ is a base for $H_i / C_{H_i}(V_i)$, we get that $g = \cap_i C_{H_i}(V_i) = 1$.

Now we turn to the reduction of Theorem 1.2 to primitive groups. Notice that N is a p-solvable group and V is the sum of at least k irreducible $\mathbb{F}_q N$-modules, so we have $|N| \le 24^{-k/3} |V|^{d-1}$ by Section 4. By the last paragraph of Section 4, we may assume that $q = 3$ (and $p = 3$). In particular, the permutation group $G/N \le S_k$ is 3-solvable, and so it does not contain any non-Abelian alternating composition factor. Now [18, Corollary 1.5] implies that $|G/N| \le 24^{(k-1)/3}$. But then $|G| = |N||G/N| \le 24^{-1/3} |V|^{d-1}$ which is exactly what we wanted.

6. Groups of semilinear transformations

In this section we reduce Theorems 1.1 and 1.2 to the case when every irreducible $\mathbb{F}_q N$-submodule of V is absolutely irreducible for any normal subgroup N of G.

For this purpose let $N \lhd G$ be a normal subgroup of G. Then V is a homogeneous $\mathbb{F}_q N$-module, so $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$, where the V_i's are isomorphic irreducible $\mathbb{F}_q N$-modules. Let $T := \mathrm{End}_{\mathbb{F}_q N}(V_1)$. Assuming that the V_i's are not absolutely irreducible, T is a proper field extension of $\mathbb{F}_q$, and

$$C_{GL(V)}(N) = \mathrm{End}_{\mathbb{F}_q N}(V) \cap GL(V) \simeq GL(k, T),$$

since $\mathrm{End}_{\mathbb{F}_q N}(V)$ is isomorphic to the matrix algebra $M_k(T)$ by [7, Theorem 1.7.5]

Furthermore, $L = Z(C_{GL(V)}(N)) \simeq Z(GL(k, T)) \simeq T^{\times}$. Now, by using L, we can extend V to a T-vector space of dimension $l := \dim_T V < \dim_{\mathbb{F}_q} V$. As

$G \leq N_{GL(V)}(L)$, in this way we get an inclusion $G \leq \Gamma L(l,T)$. We proceed by proving the following theorem.

Theorem 6.1. *For a proper field extension T of $\mathbb{F}_q$ let $G \leq \Gamma L(l,T)$ be a semilinear group acting on the $\mathbb{F}_q$-space V and let $H = G \cap GL(l,T)$. Suppose that G is p-solvable and that $b(H) \leq t(|T|)$. Then $b(G) \leq t(|T|)$.*

Proof. We modify the proof of [12, Lemma 6.1] to make it work in this more general setting.

Clearly we may assume that $|T| \geq 8$ is different from a prime. In these cases $t(|T|) = 2$.

Let u_1, u_2 be a base for H. By Corollary 3.2, we may also assume that

$$N_H(\langle u_1\rangle) \cap N_H(\langle u_1, u_2\rangle) \subseteq N_H(\langle u_2\rangle)$$

or that every non-identity element of $C_H(u_1) \cap N_H(\langle u_1, u_2\rangle)$ takes u_2 to $u_2 + u_1$. (The latter case occurs only if $p = 2$.)

For every $\alpha \in T$ let $H_\alpha = C_G(u_1) \cap C_G(u_2 + \alpha u_1) \leq G$. Our goal is to prove that $H_\alpha = 1$ for some $\alpha \in T$. If $g \in \langle \cup H_\alpha \rangle$, then $g(u_1) = u_1$ and $g(u_2) = u_2 + \delta u_1$ for some $\delta \in T$.

We claim that $|\langle \cup H_\alpha\rangle \cap H| \leq 2$. Let $h \in \langle \cup H_\alpha\rangle \cap H$. On the one hand, the action of h on V is T-linear, since $h \in H$. On the other hand, $h(u_1) = u_1$ and $h(u_2) = u_2 + \delta u_1$ for some $\delta \in T$. By our assumption above, either $h \in N_H(\langle u_2\rangle)$ and $\delta = 0$, or h is an involution and $\delta = 1$. Thus we obtain the claim since $C_H(u_1) \cap C_H(u_2) = 1$.

Let z be the generator of the group $\langle \cup H_\alpha\rangle \cap H$. This is a central element in $\langle \cup H_\alpha\rangle$. For every $g \in G$ let $\sigma_g \in \mathrm{Gal}(T|\mathbb{F}_q)$ denote the action of g on T.

Let g and h be two elements of $\langle \cup H_\alpha\rangle$. Since G/H is embedded into $\mathrm{Gal}(T|\mathbb{F}_q)$, we get $\sigma_g \neq \sigma_h$ unless $g = h$ or $g = hz$. Furthermore, a routine calculation shows that the subfields of T fixed by σ_g and σ_h are the same if and only if $\langle g\rangle = \langle h\rangle$ or $\langle g\rangle = \langle hz\rangle$.

If $g \in H_\alpha \cap H_\beta$, then $g(u_2) = u_2 + (\alpha - \alpha^{\sigma_g})u_1 = u_2 + (\beta - \beta^{\sigma_g})u_1$, so $\alpha - \beta$ is fixed by σ_g. Let $K_g = \{\alpha \in T \mid g \in H_\alpha\}$. The previous calculation shows that K_g is an additive coset of the subfield fixed by σ_g, so $|K_g| = p^d$ for some $d \mid f = \log_q |T|$. Since for any $d \mid f$ there is a unique p^d-element subfield of T, we get $|K_g| \neq |K_h|$ unless the subfields fixed by σ_g and σ_h are the same. As we have seen, this means that $\langle g\rangle = \langle h\rangle$ or $\langle g\rangle = \langle hz\rangle$. Consequently, $|K_g| \neq |K_h|$ unless $K_g = K_h$ or $K_g = K_{hz}$. Hence we get

$$|\bigcup_{g \in \cup H_\alpha \setminus \{1\}} K_g| \leq 2 \sum_{d|f, d<f} q^d \leq 2 \sum_{d<f} q^d < q^f = |T|.$$

So there is a $\gamma \in T$ which is not contained in K_g for any $g \in \cup H_\alpha \setminus \{1\}$. This exactly means that $H_\gamma = C_G(u_1) \cap C_G(u_2 + \gamma u_1) = 1$. $\square$

Using Theorem 6.1, we can assume that $G \leq GL(l,T)$. As $l = \dim_T V < \dim_{\mathbb{F}_q}(V)$, we can use induction on the dimension of V, thus $b(G) \leq 2$.

By the last paragraph of Section 4, we need not consider Theorem 1.2 here.

Hence in the following we assume that V is a direct sum of isomorphic absolutely irreducible $\mathbb{F}_q N$-modules for any $N \lhd G$.

7. Stabilizers of tensor product decompositions

Let $N \lhd G$ and let $V = V_1 \oplus \cdots \oplus V_k$ be a direct decomposition of V into isomorphic absolutely irreducible $\mathbb{F}_q N$-modules. By choosing a suitable basis in $V_1, V_2, \ldots, V_k$, we can assume that $G \le GL(n,q)$ such that any element of N is of the form $A \otimes I_k$ for some $A \in N_{V_1} \le GL(n/k,q)$. By using [14, Lemma 4.4.3(ii)] we get

$$N_{GL(n,q)}(N) = \{B \otimes C \,|\, B \in N_{GL(n/k,q)}(N_{V_1}),\ C \in GL(k,q)\}.$$

Let

$$G_1 = \{g_1 \in GL(n/k,q) \,|\, \exists g \in G, g_2 \in GL(k,q) \text{ such that } g = g_1 \otimes g_2\}.$$

We define $G_2 \le GL(k,q)$ in an analogous way. Then $G \le G_1 \otimes G_2$. Here $G/Z \simeq (G_1/Z) \times (G_2/Z)$, hence $G_1 \le GL(n/k,q)$ and $G_2 \le GL(k,q)$ are p-solvable irreducible linear groups. If $1 < k < n$, then by using induction for $G_1 \le GL(n/k,q)$ and $G_2 \le GL(k,q)$ we get $b(G_1) \le t(q)$ and $b(G_2) \le t(q)$. Furthermore $b^*(G_1) \le t(q)$ and $b^*(G_2) \le t(q)$ by Corollary 3.3. Thus [16, Lemma 3.3 (ii)] gives us

$$b(G) \le b(G_1 \otimes G_2) \le b^*(G_1 \otimes G_2) \le \\ \max(b^*(G_1), b^*(G_2)) \le t(q).$$

For the reduction of Theorem 1.2, by using induction on the dimension, we have

$$|G| \le |G_1| \cdot |G_2| \le 24^{-1/3} q^{(n/k)(d-1)} \cdot 24^{-1/3} q^{k(d-1)} \le 24^{-1/3} |V|^{d-1}.$$

Thus, from now on we can assume that for every normal subgroup $N \lhd G$ either $N \le Z$ or V is absolutely irreducible as an $\mathbb{F}_q N$-module.

8. Groups of symplectic type

From now on assume that N is a normal subgroup of G containing Z such that N/Z is a minimal normal subgroup of G/Z. Then N/Z is a direct product of isomorphic simple groups. In this section we examine the situation when N/Z is an elementary Abelian group.

If N is Abelian then it is central in G. So assume that N is non-Abelian.

If N/Z is elementary Abelian of rank at least 2, then G is of symplectic type. Such groups were examined in [12, Section 5] (see also Remark 5.20 in [12]) where it was proved that $b(G) \le 2$ unless $q \in \{3,4\}$, when $b(G) \le 3$ holds.

For the reduction of Theorem 1.2, we need only examine the case $q = 3,\ n = 2^k$. For this we can use the fact that G/N can be considered as a subgroup of the symplectic group $\mathrm{Sp}(2k,2)$. By the theorem of Pálfy [19] and Wolf [23], we may assume that G is a non-solvable (and 3-solvable) group. Thus we must have a composition factor of G (and thus of G/N) isomorphic to a Suzuki group. Since the smallest Suzuki group Suz(8) has order larger than $|\mathrm{Sp}(4,2)|$, we must have $k \ge 3$. On the other hand, since the second largest Suzuki group Suz(32) has order larger than $|\mathrm{Sp}(6,2)|$ and since Suz(8) is not a section of $\mathrm{Sp}(6,2)$ (since 13 divides the order of the first group but not the order of the second), we see that $k \ne 3$. But for $k \ge 4$ we clearly have $|G| = |N||G/N| < 2^{2k^2+3k+3} < 24^{-1/3}|V|^{d-1}$, by use of the formula for the order of $\mathrm{Sp}(2k,2)$.

9. Tensor product actions

Now let N/Z be a direct product of $t \geq 2$ isomorphic non-Abelian simple groups. Then $N = L_1 \star L_2 \star \cdots \star L_t$ is a central product of isomorphic groups such that for every $1 \leq i \leq t$ we have $Z \leq L_i$, L_i/Z is simple. Furthermore, conjugation by elements of G permutes the subgroups $L_1, L_2, \ldots, L_t$ in a transitive way. By choosing an irreducible $\mathbb{F}_q L_1$-module $V_1 \leq V$, and a set of coset representatives $g_1 = 1, g_2, \ldots, g_t \in G$ of $G_1 = N_G(V_1)$ such that $L_i = g_i L_1 g_i^{-1}$, we get that $V_i := g_i V_1$ is an absolutely irreducible $\mathbb{F}_q L_i$-module for each $1 \leq i \leq t$. Now, $V \simeq V_1 \otimes V_2 \otimes \cdots \otimes V_t$ and G permutes the factors of this tensor product. It follows that G is embedded into the central wreath product $G_1 \wr_c S_t$ defined to be a split extension of the base group $\underbrace{G_1 \otimes G_1 \otimes \cdots \otimes G_1}_{t \text{ factors}}$ by S_t. Clearly $G_1 \leq GL(V_1)$ is a p-solvable irreducible linear group. Thus $b(G_1) \leq t(q)$ and $b^*(G_1) \leq t(q)$ by induction on the dimension m of V_1 and by Corollary 3.3.

First let $q \geq 5$. Then $t(q) = 2$. Thus $b(G) \leq 2$ follows from [12, Theorem 3.6] unless $(m,t) = (2,2)$. In case $(m,t) = (2,2)$, that is, $G \leq G_1 \wr_c S_2 \leq GL(4,q)$ for some p-solvable group $G_1 \leq GL(2,q)$ let $x_1, y_1 \in V_1$ be a basis of V_1 satisfying either $N_{G_1}(\langle x_1 \rangle) \subseteq N_{G_1}(\langle y_1 \rangle)$ or the property that every non-identity element of $C_{G_1}(x_1)$ takes y_1 to $y_1 + x_1$. (Such a basis exists by Theorem 3.1.) We claim that if $\alpha \in \mathbb{F}_q \setminus \{0,1\}$ then $x_1 \otimes x_1,\ y_1 \otimes (y_1 + \alpha x_1)$ is a base for $G_1 \wr_c S_2 \geq G$. Indeed, let $g = (A \otimes B)\sigma \in G_1 \wr_c S_2$ with $A, B \in G_1,\ \sigma \in S_2$ fixing these two vectors. Then $g(x_1 \otimes x_1) = x_1 \otimes x_1$ implies that $Ax_1 = \lambda x_1,\ Bx_1 = \lambda^{-1} x_1$ for some $\lambda \in \mathbb{F}_q^\times$. If $N_{G_1}(\langle x_1 \rangle) \subseteq N_{G_1}(\langle y_1 \rangle)$, then $Ay_1 = ay_1,\ By_1 = by_1$ for some $a, b \in \mathbb{F}_q^\times$. Hence

$$y_1 \otimes (y_1 + \alpha x_1) = g(y_1 \otimes (y_1 + \alpha x_1)) = ab y_1 \otimes y_1 + \alpha a \lambda^{-1} (y_1 \otimes x_1)^\sigma.$$

Comparing the coefficients of $y_1 \otimes y_1$ and $y_1 \otimes x_1$ in the above equality we get $ab = a\lambda^{-1} = 1$ and $\sigma = 1$. So, $A = \lambda I,\ B = \lambda^{-1} I$ and $g = 1$, as claimed. Similarly, if every non-identity element of $C_{G_1}(x_1)$ takes y_1 to $y_1 + x_1$, then by multiplying A with λ^{-1} and B with λ, we can assume that $\lambda = 1$. Then for some $\varepsilon_a, \varepsilon_b \in \{0,1\}$ we have

$$y_1 \otimes (y_1 + \alpha x_1) = g(y_1 \otimes (y_1 + \alpha x_1)) = \Big((y_1 + \varepsilon_a x_1) \otimes (y_1 + (\alpha + \varepsilon_b) x_1) \Big)^\sigma.$$

Comparing the coefficients of $x_1 \otimes x_1, x_1 \otimes y_1$ and $y_1 \otimes x_1$ we get $\varepsilon_a = \varepsilon_b = 0,\ \sigma = 1$, so $g = 1$ follows.

Now, let $q = 3$. Let $x_1, y_1, z_1 \in V_1$ be a strong base for G_1. Then the stabilizer of $\underbrace{x_1 \otimes x_1 \otimes \cdots \otimes x_1}_{t \text{ factors}} \in V$ is of the form $H = H_1 \wr_c S_t$, where $y_1, z_1 \in V_1$ is a strong base for $H_1 = N_{G_1}(x_1)$, so $b^*(H_1) \leq 2$. If $(m,t) \neq (2,2)$ then $b(H) \leq 2$ by [12, Theorem 3.6], which results in $b(G) \leq 3$. Finally, let $(m,t) = (2,2)$. By choosing a basis $x_1, y_1 \in V_1$, it is easy to see that $x_1 \otimes x_1, y_1 \otimes y_1, x_1 \otimes y_1 \in V$ is a base for $GL(V_1) \wr_c S_2 \geq G$.

As for the order of G notice that $G \leq G_1 \wr_c S$ where $S \leq S_t$ is a 3-solvable group. Thus by induction and by [18, Corollary 1.5] we have

$$|G| \leq |G_1|^t |S| \leq 24^{-t/3} |V_1|^{(d-1)t} 24^{(t-1)/3} = 24^{-1/3} |V|^{d-1}.$$

10. Almost quasisimple groups

Finally, let $Z \leq N \lhd G$ be such that N/Z is a non-Abelian simple group. Let $N_1 = [N, N] \lhd G$ and let V_1 be an irreducible $\mathbb{F}_p N_1$-submodule of V and $G_1 = \{g \in G \,|\, g(V_1) = V_1\}$ be the stabilizer of V_1. By using the same argument as in the last paragraph of [12, Page 29] we get that G_1 is included in $GL(V_1)$ and we have a chain of subgroups $N_1 \lhd G_1 \leq GL(V_1)$ where G_1 is p-solvable, N_1 is quasisimple and V_1 is irreducible as an $\mathbb{F}_p N_1$-module.

Suppose that $b(G_1) \leq 2$ in the action of G_1 on V_1, that is, there exist $x, y \in V_1 \leq V$ such that $C_{G_1}(x) \cap C_{G_1}(y) = 1$. For any element $g \in G$ with $g(x) = x$ we have that $N_1 x = \{nx \,|\, n \in N_1\}$ is a g-invariant subset. As the $\mathbb{F}_p$-subspace generated by $N_1 x$ is exactly V_1, we get that $g \in G_1$. This proves that $C_G(x) \cap C_G(y) = C_{G_1}(x) \cap C_{G_1}(y) = 1$. Thus $b(G) \leq 2$.

Hence if we manage to show that $b(G_1) \leq 2$ then we are finished with the proofs of both Theorems 1.1 and 1.2.

So assume that $G = G_1$, $N = N_1$, and $V = V_1$. By the first three paragraphs of this section, we have that $q = p$. To summarize, $G \leq GL(V)$ is a group having a quasisimple irreducible normal subgroup N and $Z \leq G$.

We can assume that G/Z is almost simple. For this it is sufficient to see that N/Z is the unique minimal normal subgroup of G/Z. For let M/Z be another minimal normal subgroup of G/Z. By Section 8, we may assume that M/Z is non-Abelian. Furthermore the group MN is a central product and so $[M, N] = 1$. But this is impossible since the centralizer of N in G must be Abelian.

Lemma 10.1. *If N has a regular orbit on V then $b(G) \leq 2$.*

Proof. Since N is normal in G, a regular N-orbit Δ containing a given vector v is a block of imprimitivity inside the G-orbit containing v. Hence the group $C_G(v)N$ is transitive on Δ and N is regular on Δ. Thus for every $h \in C_G(v)$ the number $|\mathrm{fix}(h)|$ of fixed points of h on Δ is $|C_N(h)|$. To prove that G has a base of size at most 2 on V, it is sufficient to see that there exists a vector w in Δ that is not fixed by any non-trivial element of $C_G(v)$.

First notice that if $N/Z(N)$ is isomorphic to the non-Abelian finite simple group S then $|C_G(v)| \leq |\mathrm{Out}(S)| < m(S)$ where $m(S)$ is the minimal index of a proper subgroup of S. This latter inequality follows from [1, Lemma 2.7 (i)].

But

$$\sum |\mathrm{fix}(h)| = \sum |C_N(h)| < |C_G(v)| \cdot \frac{|N|}{m(S)} < |N|$$

where the sums are over all non-identity elements h in $C_G(v)$. This completes the proof of the lemma. □

By Lemma 10.1, in the following we may assume that N does not have a regular orbit on V. Our final theorem finishes the proofs of Theorems 1.1 and 1.2.

Theorem 10.2. *Under the current assumptions G is a p'-group and $b(G) \leq 2$.*

Proof. By using Goodwin's theorem [11, Theorem 1], Köhler and Pahlings [15, Theorem 2.2] gave a complete list of (irreducible) quasisimple p'-groups N such that N does not have a regular orbit on V. In all these exceptional cases, when N/Z is simple, $|\mathrm{Out}(N/Z)|$ is divisible by no prime larger than 3 while p is always at least 5. So G itself is a p'-group. But then G admits a base of size 2 on V by [12, Theorem 4.4]. □

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