The Maximal Defect k-Subgroups of Semigroups of Graphs and Digraphs

Károly Podoski

3rd BIOMICS Workshop
Passau, 9 February 2016

SEVENTH FRAMEWORK PROGRAMME

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Graphs and Digraphs

- *Digraph*: $\Gamma = (V, E)$, with *vertices* V, *edges* $E \subseteq V \times V$.
- The *reverse edge* to $e = (x, y) \in E$ is $\bar{e} = (y, x)$
- *(Undirected) Graph*: E is symmetric
- No self-loops: $(x, x) \notin E$.

Spectrum of Graph

Algebraic graph theory

- *adjacency matrix* $A = A(\Gamma)$ of $\Gamma = (V, E)$ is a matrix of 0's and 1's with $A_{x,y} = 1$ iff $(x, y) \in E$.
- The multiset of eigenvalues of A is called the *spectrum* of A.

Theorem

The spectrum of graphs and digraphs is invariant for isomorphic (di)graphs.

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Example (Co-spectral Graphs)

$K_{1,4}$ and $C_4 \sqcup K_1$ have same spectrum. $\Longrightarrow$ not complete invariant

Elementary Collapsings of Directed Edges

Digraph $\Gamma = (V, E)$

Elementary collapsing

edge $e = (x, y) \longrightarrow$ function $T_{x,y} \colon V \to V$

$$T_{x,y}(v) = \begin{cases} y & \text{if } v = x, \\ v & \text{otherwise.} \end{cases}$$

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Example

$$\begin{aligned} T_{(1,2)} \colon 1 &\to 2 \\ 2 &\to 2 \\ 3 &\to 3 \\ 4 &\to 4 \end{aligned}$$

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Example

$$T_{(1,2)}$$

Motivation: Biochemical Reactions

Biochemical transitions are modelled as products of commuting elementary collapsings, $f = \prod T_{a,b}$, where $T_{x,y}$ and $T_{y,z}$ do not both occur among the $T_{a,b}$ for any x, y, and z.

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Definition

We say a mapping $f : V \to V$ has *defect* k if $|f(V)| = |V| - k$.

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f has defect $k \implies fg$ and gf have defect at least k.

Composition of Elementary Collapsings

Example (Γ is the 3-cycle with edges $(1,2),(2,3),(3,1)$)

compose functions from left to right

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& 2 \to 2 \to 3 & & 2 \to 3 \to 3 \\
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Composition of Elementary Collapsings

Definition

The *transformation semigroup* of a digraph $\Gamma = (V, E)$ to be the collection of transformations of V generated by the T_e $(e \in E)$.

$$S(\Gamma) = \langle T_{x,y} \in V^V \,|\, (x, y) \text{ is an edge of } \Gamma \rangle.$$

This is also called the semigroup of flows on Γ.

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$$T_{1,2}T_{2,3} : \begin{array}{l} 1 \to 2 \to 3 \\ 2 \to 2 \to 3 \\ 3 \to 3 \to 3 \end{array} \qquad T_{2,3}T_{1,2} : \begin{array}{l} 1 \to 1 \to 2 \\ 2 \to 3 \to 3 \\ 3 \to 3 \to 3 \end{array}$$

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Theorem

$S(\Gamma)$ *is an invariant for graphs and for digraphs.*

Proof.

Isomorphic (di)graphs obviously have isomorphic semigroups of flows. □

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Theorem (Nehaniv–Rhodes)

$S(\Gamma)$ is a complete algebraic invariant for undirected graphs.

Semigroup of Flows on a Graph

Example

There exist non-isomorphic digraphs with the same flow semigroup. 3-cyclic and 3-cycle with one reverse edge .

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Now $f^2 = T_{3,2}$, corresponding to an edge $(3,2)$.

Complete Invariance for Graphs

Lemma

If $T_{x,y} = T_{x_1,y_1} \cdots T_{x_k,y_k}$*, then* $T_{x,y}$ *or* $T_{y,x}$ *appears among* T_{x_i,y_i}*.*

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Corollary

Let Γ *be a digraph.*
If $T_{x,y} \in S(\Gamma)$ *then either* $e = (x, y)$ *or* $\bar{e} = (y, x)$ *is an edge of* Γ*.*

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$\Longrightarrow$ $S(\Gamma)$ is a complete invariant for graphs.

Reversing Edges in Directed Cycles

Δ *directed n-cycle*: n nodes, edges $(1, 2), \ldots, (n-1, n)$, $(n, 1)$.

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So, $S(\Delta \cup \{(1,n)\}) = S(\Delta)$.

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Reversing Edges in Directed Cycles

Theorem

Digraph Γ. If e is an edge in a simple directed cycle in Γ, then $S(\Gamma) = S(\Gamma \cup \{\bar{e}\})$. $\Longrightarrow$ Enough to consider undirected graphs on strongly connected components.

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W defect 1: permutes $n-1$ vertices cyclically

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Defect k Permutator Groups

Permutator Group

- If $e^2 = e \colon V \to V$ is an idempotent $S(\Gamma)$.
  Let $X_e = V \cdot e = \{v \cdot e \,|\, v \in V\}$
  Let $G_e =$ (unique) maximal subgroup of $S(\Gamma)$ containing e.
- (X_e, G_e) (faithful) *permutator group* of subset X_e, consists of defect k where $k = |V| - |X_e|$.

Theorem

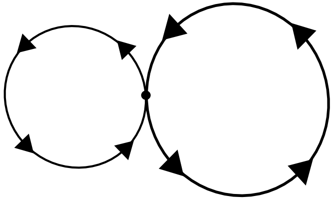
The defect k permutator groups (up to isomorphism) are invariants for digraphs.

Some Defect 1 Permutator Groups

Example

- Cycle graph with n nodes: Defect 1 group: cyclic C_{n-1}

Some Defect 1 Permutator Groups



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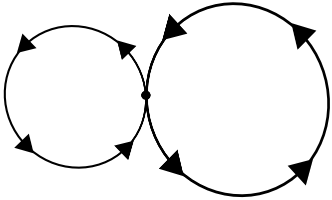

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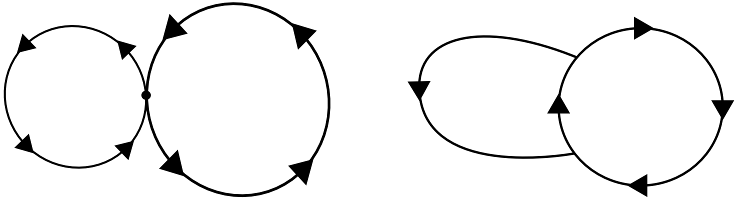

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- Overlapping Cycles:

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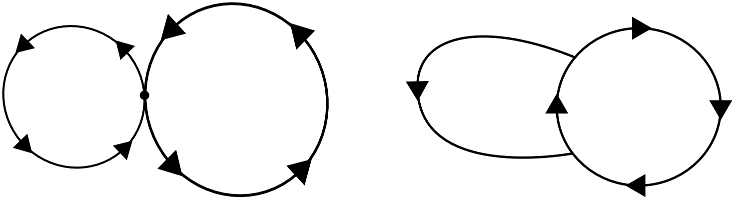

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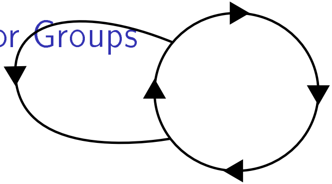
- Overlapping Cycles: Defect 1 group: S_{n-1} or A_{n-1} or S_5

Defect 1 Permutator Groups

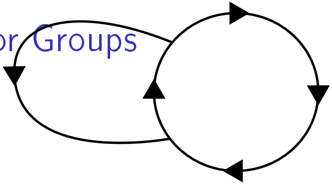


Theorem

$$T_{k,l,m} = \langle x, y \rangle, n = k + l + m,$$
$$x = (a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l),$$
$$y = (a_1, a_2, \ldots, a_k, c_1, c_2, \ldots, c_m),$$

1. *If* $k = 0$ *then* $T_{0,l,m} \simeq C_l \times C_m$*;*
2. *if* $k \geq 1$*,* $k + l$ *and* $k + m$ *are both odd, then* $T_{k,l,m} = A_n$*;*
3. $T_{3,2,1} \simeq T_{2,2,2} \simeq T_{3,1,2} \simeq S_5$*, and this is a 3-transitive action of* S_5 *on 6 elements;*
4. $T_{k,l,m} = S_n$*, otherwise.*

Defect 1 Permutator Groups



- ear decomposition

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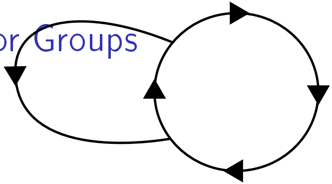
Defect 1 Permutator Groups



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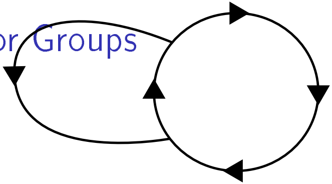
Defect 1 Permutator Groups



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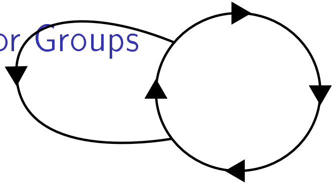

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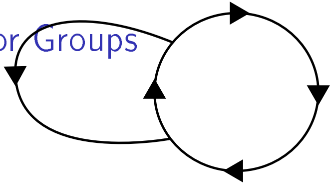

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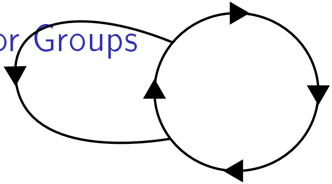

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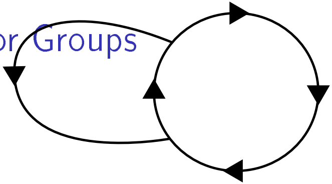

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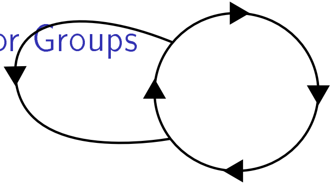

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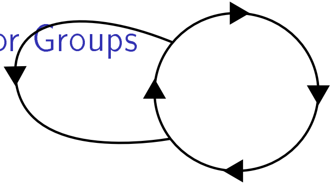

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- done for all 2-vertex connected graphs
- direct products of defect 1 groups of biconnected components

Main Theorem (Horváth, Nehaniv, Podoski, 2015)

Graph Γ, 2-edge connected, not a cycle.
Then maximal defect k groups are all isomorphic and

- Defect 1: Direct product of cyclic, alternating and symmetric groups of various orders, (corresponding to the biconnected components)
- Defect $k \geq 2$, no bridge with $\geq k$ nodes $\Longrightarrow S_{n-k}$
- Defect $k \geq 2$, bridges with $\geq k$ nodes $\Longrightarrow$ direct product of symmetric groups (corresponding to the biconnected components with $\leq k-1$ long bridges)

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