Multiplicative loops of 2-dimensional topological quasifields

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The subject of this talk comes from the theory of affine planes and the algebraic structure coordinatizing them. Classical examples are the Desarguesian planes, they are coordinatized by skewfields. Here we deal with the class of quasifields. This includes the class of fields and the class of division algebras. Both of these algebraic structures are determined by the existence of both distributive laws. But already for division algebras a complete classification without the assumption of locally compactness is impossible. The connected locally compact fields are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$. The octonions $\mathbb{O}$ over $\mathbb{R}$ are the only non-associative locally compact alternative algebra.

Definition

A (left) quasifield is an algebraic structure $(Q, +, \cdot)$ *such that* $(Q, +)$ *is an abelian group with neutral element* 0*,* $(Q \setminus \{0\}, \cdot)$ *is a loop with identity element* 1*, i. e. the equations* $a \cdot y = b$ *and* $x \cdot a = b$ *have precisely one solution for given* $a, b \in Q \setminus \{0\}$*, and between these operations the (left) distributive law* $x \cdot (y + z) = x \cdot y + x \cdot z$ *holds. If for any given* $a, b, c \in Q$ *with* $a \neq b$ *the equation* $a \cdot x - b \cdot x = c$ *has precisely one solution, then* Q *is called planar. The mappings* $\lambda_a : Q \to Q, x \mapsto a \cdot x$*,* $\rho_a : Q \to Q, x \mapsto x \cdot a$*,* $a \in Q$*, are the left translations, respectively the right translations of* Q*. Since* $Q^* = Q \setminus \{0\}$ *is a loop the left and right translations with the elements* $a \in Q \setminus \{0\}$ *are bijections of* Q*.*

The quasifields are intensively studied in the finite, respectively in the locally compact case, which are the coordinate domains of finite, respectively locally compact translation planes.

If W is a line of an affine plane $\mathcal{A}$ and w denotes the improper point of W, then $W \cup \{w\}$ is a line of the projective extension of $\mathcal{A}$. The improper line of $\mathcal{A}$ is denoted by L_∞. The group of all collineations with axis L_∞ are called dilatations of $\mathcal{A}$ and the dilatations whose center is on L_∞ are called translations.

Definition

A translation plane is an affine plane $\mathcal{A}$ if the translation group of $\mathcal{A}$ operates transitively on the point set of $\mathcal{A}$.

A quasifield Q gives rise to a translation plane $\mathcal{A}$ by setting $Q \times Q$ as the point set of $\mathcal{A}$ and defining the lines by equations $X = c$ and $Y = mX + b$. The translations have the form $(x, y) \mapsto (x + u, y + v)$ with fixed $u, v \in Q$.

Definition

A subfield K_r of a quasifield Q is called the kernel of Q if for all $k \in K_r$ and $x, y \in Q$ one has

$$(x+y)\cdot k = x\cdot k + y\cdot k \text{ and } (x\cdot y)\cdot k = x\cdot(y\cdot k).$$

The quasifield Q is a right vector space over K_r. Moreover, for all $a \in Q$ the map $\lambda_a : Q \to Q, x \mapsto ax$ is K_r-linear.

Definition

A locally compact connected topological quasifield is a locally compact connected topological space Q such that $(Q,+)$ is a topological group, $(Q\setminus\{0\},\cdot)$ is a topological loop, the multiplication $\cdot : Q\times Q \to Q$ is continuous and the mappings $\lambda_a : x \mapsto a\cdot x$ and $\rho_a : x \mapsto x\cdot a$ with $0 \neq a \in Q$ are homeomorphisms of Q.

Vector space structure of locally compact connected quasifields

The kernel K_r is the field of the real numbers and the dimension of Q over $\mathbb{R}$ is 1, 2, 4 or 8.
The kernel K_r is the field of the complex numbers and the dimension of Q over $\mathbb{C}$ is 1 or 2.
Q is the skewfield of quaternions.

(H. Salzmann: Topological planes, 7.12, Adv. Math. 2, 1967)
Every locally compact connected quasifield is planar since it has finite dimension over its kernel.

In this talk we deal with locally compact 2-dimensional topological connected quasifields. Their kernel is $\mathbb{R}$, $(Q,+)$ is the vector group $\mathbb{R}^2$ and the multiplicative loop $(Q\setminus\{0\},\cdot)$ is homeomorphic to $\mathbb{R}\times S^1$, where S^1 is the circle.
They coordinatize locally compact 4-dimensional topological translation planes. The classification of these topological translation planes $\mathcal{A}$ can be obtained by using spreads.
Every translation plane can be obtained from a spread of a suitable vector space V.

Definition

Let V be a vector space over a skewfield. A collection $\mathcal{B}$ of subspaces of V with $|\mathcal{B}|\geq 3$ is called a spread of V if for any two different elements U_1, $U_2\in\mathcal{B}$ we have $V=U_1\oplus U_2$ and every vector of V is contained in an element of $\mathcal{B}$.

The points of $\mathcal{A}$ can be identified with the elements of V. The lines through the origin form a spread $\mathcal{B}$ of V, the other lines are cosets of the elements of $\mathcal{B}$.

If S and W are different subspaces of the spread $\mathcal{B}$, then V can be coordinatized in such a way that $S = \{0\} \times X$ and $W = X \times \{0\}$. Any spread of $V = X \times X$ can be described by a collection $\mathcal{M}$ of linear mappings $X \to X$ satisfying the following conditions:

(M_1) For any $\omega_1 \neq \omega_2 \in \mathcal{M}$ the mapping $\omega_1 - \omega_2$ is bijective.

(M_2) For all $x \in X \setminus \{0\}$ the mapping $\phi_x : \mathcal{M} \to X : \omega \mapsto \omega(x)$ is surjective.

Namely, if $\mathcal{M}$ is a collection of linear mappings satisfying (M_1) and (M_2), then the sets $U_\omega = \{(x, \omega(x)), x \in X\}$ and $\{0\} \times X$ yield a spread of $V = X \times X$. Conversely, every component $U \in \mathcal{B} \setminus \{S\}$ of V is the graph of a linear mapping $\omega_U : W \to S$ and the set of ω_U gives a collection $\mathcal{M}$ of linear mappings of X satisfying (M_1) and (M_2) (cf. N. Knarr: Translation Planes). The mapping ω_W is the zero mapping. For this reason any collection $\mathcal{M}$ of linear mappings of X which satisfy (M_1) and (M_2) is called a spread set of X.

2-dimensional locally compact connected topological quasifields

Let e_1 be the identity element of the multiplicative loop $Q^* = (Q \setminus \{0\}, \cdot)$ of Q, which generates the kernel $K_r = \mathbb{R}$ of Q as a vector space and let $B = \{e_1, e_2\}$ be a basis of the right vector space Q over K_r. Once we fix B, we identify Q with the vector space of pairs $(x, y)^t \in \mathbb{R}^2$ and K_r with the subspace of pairs $(x, 0)^t$. The element $(1, 0)^t$ is the identity element of the multiplicative loop Q^* of Q.

Fixing this basis the $\mathbb{R}$-linear maps λ_a can be written as 2×2 matrices over $\mathbb{R}$. For all $a, b \in Q$, $a \neq b$, the mapping $\lambda_a - \lambda_b : Q \to Q, x \mapsto a \cdot x - b \cdot x$ is bijective and for all $x \in Q \setminus \{0\}$ the map $\lambda_x : Q \to Q, m \mapsto x \cdot m$ is surjective. Hence to the set of the linear maps λ_a yields a spread set of $X = \mathbb{R}^2$.

As every translation plane can be coordinatized by a quasifield and a quasifield contains 0 and 1, the associated spread set contains the zero endomorphism and the identity map. This is not true for arbitrary spread sets $\mathcal{M}$, but if $\omega_0, \omega_1 \in \mathcal{M}$ are distinct, then $\mathcal{M}' = \{(\omega - \omega_0)(\omega_1 - \omega_0)^{-1},\ \omega \in \mathcal{M}\}$ is a normalized spread set of X, i.e. a spread set which contains the zero and the identity map. The translation planes obtained from $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic. Any spread gives the lines through the origin and hence the multiplication in a 2-dimensional quasifield Q coordinatizing the plane $\mathcal{A}$.
Hence the set $\{\lambda_a; a \in Q\}$ is a normalized spread set of the vectorspace $X = \mathbb{R}^2$. Since the set of vectors $(x, y)^t, x, y \in \mathbb{R}$ consists of all vectors of X, there exists a unique left translation λ_a such that its representation as 2×2 matrix has $(x, y)^t$ as the first column.

The algebraic structure of the proper multiplicative loop Q^* of Q

The group topologically generated by the left translations of Q^* is the connected component of $\mathrm{GL}_2(\mathbb{R})$, the group topologically generated by the right translations of Q^* is an infinite-dimensional Lie group (cf. P. T. Nagy and K. Strambach: Loops in Group Theory and Lie Theory) and any locally compact 2-dimensional semifield (i.e. in Q also the right distributive law holds) is the field of complex numbers (cf. P. Plaumann and K. Strambach: Zweidimensionale Quasialgebren mit Nullteilern. Aequationes Math.).

Definition

A loop L is called topological, if it is a topological space and the binary operations $(a,b) \mapsto a \cdot b$, $(a,b) \mapsto b/a$, $(a,b) \mapsto a\backslash b : L \times L \to L$ are continuous. If L is a connected differentiable manifold such that the above mappings are continuously differentiable, then L is a $\mathcal{C}^1$-differentiable loop.

Every topological connected loop L having a Lie group G as the group topologically generated by the left translations of L corresponds to a sharply transitive continuous section $\sigma : G/H \to G$, where $G/H = \{xH | x \in G\}$ consists of the left cosets of the stabilizer H of $1 \in L$ such that $\sigma(H) = 1_G$ and $\sigma(G/H)$ generates G. The section σ is sharply transitive if the image $\sigma(G/H)$ acts sharply transitively on the factor space G/H, i.e. for given left cosets xH, yH there exists precisely one $z \in \sigma(G/H)$ which satisfies the equation $zxH = yH$.

Since the group G topologically generated by the left translations of Q^* is the connected component of the group $GL_2(\mathbb{R})$, $\dim\, Q^* = 2$ and the stabilizer H of $e \in Q^*$ in G does not contain any normal subgroup $\neq 1$ of G we assume that H is the subgroup $\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}$ isomorphic to $\mathcal{L}_2$.

The elements g of G have a unique decomposition as the product

$$g = \begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} \begin{pmatrix} k & l \\ 0 & k^{-1} \end{pmatrix}$$

with suitable elements $u > 0,\ k > 0,\ l \in \mathbb{R},\ t \in [0, 2\pi)$.

Hence the loop Q^* homeomorphic to $\mathbb{R} \times S^1$ corresponds to a continuous section $\sigma : G/H \to G$;

$$(1) \qquad \begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} H \mapsto \begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} \cdot \begin{pmatrix} a(u,t) & b(u,t) \\ \mathbf{0} & a^{-\mathbf{1}}(u,t) \end{pmatrix},$$

where the pair of continuous functions $a(u,t), b(u,t) : \mathbb{R}_{>0} \times [0, 2\pi) \to \mathbb{R}$, where $\mathbb{R}_{>0}$ is the set of positive numbers, satisfies the following conditions:

$$a(u,t) > 0, \quad a(1,0) = 1, \quad b(1,0) = 0.$$

As Q is a left quasifield, any $(x,y)^t \in Q^*$ induces a linear transformation $M(x,y) \in \sigma(G/H)$. More precisely

$$(2) \quad \begin{pmatrix} x \\ y \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix} = M_{(x,y)} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} r\cos\varphi & r\sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{pmatrix} \cdot \begin{pmatrix} a(r,\varphi) & b(r,\varphi) \\ \mathbf{0} & a^{-\mathbf{1}}(r,\varphi) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

where $x = r\cos(\varphi)a(r,\varphi)$, $y = -r\sin(\varphi)a(r,\varphi)$.

As the kernel K_r consists of pairs $(0,0)^t$ and $(r\cos(k\pi)a(r,k\pi),0)^t$, $r>0$, $k\in\{0,1\}$ such that the matrices $M(r\cos(k\pi)a(r,k\pi),0)$ of K_r have the form

$$\begin{pmatrix} r\cos(k\pi)a(r,k\pi) & r\cos(k\pi)b(r,k\pi) \\ 0 & r\cos(k\pi)a^{-1}(r,k\pi) \end{pmatrix}.$$

As the identity $e\in Q^*$ is $(1,0)^T$ the corresponding matrix is I. Since to each real number $r\cos(k\pi)a(r,k\pi)$ belongs precisely one matrix $M(r\cos(k\pi)a(r,k\pi),0)$, the functions $f_1(r)=ra(r,0)$, $f_2(r)=-ra(r,\pi)$ are strictly monotone. If the functions $a(r,0)$, $a(r,\pi)$ are differentiable, then for every $r>0$ the derivatives $[ln(a(r,k\pi))]'$, $k\in\{0,1\}$ is always greater or smaller than $-r^{-1}$.

The section σ given by (1) is sharply transitive precisely if for all pairs (u_1, t_1), (u_2, t_2) in $\mathbb{R}_{>0} \times [0, 2\pi)$ there exists precisely one $(u, t) \in \mathbb{R}_{>0} \times [0, 2\pi)$ and $k > 0$, $l \in \mathbb{R}$ such that

$$\begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} \cdot \begin{pmatrix} a(u,t) & b(u,t) \\ 0 & a^{-1}(u,t) \end{pmatrix} \cdot \begin{pmatrix} u_1\cos t_1 & u_1\sin t_1 \\ -u_1\sin t_1 & u_1\cos t_1 \end{pmatrix} =$$

(3)
$$\begin{pmatrix} u_2\cos t_2 & u_2\sin t_2 \\ -u_2\sin t_2 & u_2\cos t_2 \end{pmatrix} \begin{pmatrix} k & l \\ 0 & k^{-1} \end{pmatrix}.$$

We get that $u = u_1^{-1} u_2$. Therefore the system (3) of equations is uniquely solvable if and only if for any fixed $u > 0$ the mapping

(4)
$$\sigma_u : \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} H \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \cdot \begin{pmatrix} a(u,t) & b(u,t) \\ 0 & a^{-1}(u,t) \end{pmatrix}$$

determines a quasigroup F_u homeomorphic to S^1. F_u has the right identity $(u, 0)^t$. The quasigroup F_u is a loop, i.e. $(u, 0)^t$ is the left identity of F_u, if and only if $b(u, 0) = 0$ for all $u > 0$.

Theorem

Let Q^* be the $\mathcal{C}^1$-differentiable multiplicative loop of a locally compact 2-dimensional connected quasifield Q. Then Q^* is diffeomorphic to $S^1 \times \mathbb{R}$ and belongs to a $\mathcal{C}^1$-differentiable sharply transitive section σ of the form

$$\begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} H \mapsto \begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} \cdot \begin{pmatrix} a(u,t) & b(u,t) \\ \mathbf{0} & a^{-\mathbf{1}}(u,t) \end{pmatrix},$$

with $b(u,0) = 0$ for all $u > 0$ if and only if for each fixed $u > 0$ the continuously differentiable functions $\bar{a}_u(t) := a(u,0)a^{-1}(u,t)$, $\bar{b}_u(t) := -b(u,t)$ satisfies the differential inequalities

$$\bar{a}'^2_u(t) + \bar{b}_u(t)\bar{a}'_u(t) + \bar{b}'_u(t)\bar{a}_u(t) - \bar{a}^2_u(t) < 0$$

$$(5) \qquad \bar{b}'_u(0) < 1 - \bar{a}'^2_u(0).$$

The above differential inequalities (5) were found by P.T. Nagy and K. Strambach: Loops in Group Theory and Lie Theory.

Proposition

Let Q^* be the $\mathcal{C}^1$-differentiable multiplicative loop of a locally compact 2-dimensional connected quasifield Q. Assume that for each fixed $u > 0$ the function $a_u(t) := a^{-1}(u,0)a(u,t)$ is the constant function 1 and $b(u,0) = 0$ for all $u > 0$. Then Q^* belongs to a $\mathcal{C}^1$-differentiable sharply transitive section σ of the form (1) if and only if for each fixed $u > 0$ one has $b_u(t) := b(u,t) > -t$ for all $0 < t < 2\pi$.

Proposition

Let Q^* be the $\mathcal{C}^1$-differentiable multiplicative loop of a locally compact 2-dimensional connected quasifield Q. Assume that for each fixed $u > 0$ the function $b_u(t) := b(u,t)$ is the constant function 0. Then Q^* belongs to a $\mathcal{C}^1$-differentiable sharply transitive section σ of the form (1) if and only if for each fixed $u > 0$ one has $e^{-t} < a_u(t) = a^{-1}(u,0)a(u,t) < e^t$ for all $0 < t < 2\pi$.

For the solution of these differential inequalities one can use Fourier series (cf. A. Figula, K. Strambach: Loops on spheres having a compact-free inner mapping group).

Definition

Let $\mathcal{F}$ be the set of series

$$a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt), \quad t \in \mathbb{R},$$

such that

$$1 - a_0 = \sum_{k=1}^{\infty} \frac{a_k + kb_k}{1 + k^2},$$

$$a_0 > \sum_{k=1}^{\infty} \frac{ka_k - b_k}{1 + k^2} \sin kt - \frac{a_k + kb_k}{1 + k^2} \cos kt \quad \textit{for all} \quad t \in [0, 2\pi],$$

$$2a_0 \geq \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \frac{k^2 - 1}{k^2 + 1}.$$

The differential inequalities (5) are satisfied precisely if for each fixed $u > 0$ the function $a_u^{-1}(t) := a(u,0)a^{-1}(u,t)$ has the shape

$$a_u^{-1}(t) = e^t(1 - \int_0^t R(s)e^{-s}\, ds)$$

where $R(s)$ is a continuous function the Fourier series of which is contained in the set $\mathcal{F}$ of Definition and $b_u(t) := b(u,t)$ is a periodic $\mathcal{C}^1$-differentiable function with $b_u(0) = b_u(2\pi) = 0$ such that

$$b_u(t) > -a_u(t) \int_0^t \frac{(a_u^2(s) - a_u'^2(s))}{a_u^4(s)}\, ds \quad \text{for} \quad \text{all} \quad t \in (0, 2\pi).$$

Proposition

Let

$$(6)\quad \begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} H \mapsto \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(\mathbf{1}, t) & b(u, t) \\ 0 & a^{-\mathbf{1}}(\mathbf{1}, t) \end{pmatrix},$$

$u > 0$, $t \in [0, 2\pi)$ *with* $b(u, 0) = 0$ *for all* $u > 0$ *be a section belonging to a multiplicative loop* Q^* *of a locally compact 2-dimensional connected topological quasifield* Q*. Then* Q^* *contains for any* $u > 0$ *a 1-dimensional compact subloop.*

The image of the section (6) acts sharply transitively on the point set $\mathbb{R}^2 \setminus \{(0,0)^t\}$. Since the subgroup $\left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, u > 0 \right\}$ leaves any line through $(0,0)^t$ fixed, the subset

(7)

$$\mathcal{T} = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1,t) & b(u,t) \\ 0 & a^{-1}(1,t) \end{pmatrix}, t \in [0, 2\pi) \right\}$$

acts sharply transitively on the oriented lines through $(0,0)^t$ for any $u > 0$. Therefore $\mathcal{T}$ corresponds to a 1-dimensional compact loop T.

Definition

The kernel of a homomorphism $\alpha : (L, \cdot) \to (L', *)$ of a loop L into a loop L' is a normal subloop N of L. We call a locally compact connected loop quasi-simple if it contains no normal subloop of positive dimension.

Proposition

Let Q^* be the multiplicative loop of a locally compact connected topological quasifield Q of dimension 2 containing a 1-dimensional compact normal subloop. The quasifield Q is the field $\mathbb{C}$ of complex numbers if and only if $\mathcal{T}$ is a normal subset in the set of all left translations of Q^*.

If the set $\mathcal{T}$ is a normal subset in the set of the left translations of a proper loop Q^*, then it is normal in the connected component $\mathrm{GL}_2^+(\mathbb{R})$ of the group $\mathrm{GL}_2(\mathbb{R})$ because $\mathrm{GL}_2^+(\mathbb{R})$ is the group topologically generated by the left translations of Q^*. If $\mathcal{T}$ is normal in $\mathrm{GL}_2^+(\mathbb{R})$, then for all $\varphi \in [0, 2\pi)$ one has $a(1, t) = 1$ and $b(1, t) = 0$ or equivalently $\mathcal{T} = \mathrm{SO}_2(\mathbb{R})$. But the compact group $\mathrm{SO}_2(\mathbb{R})$ is not normal in $\mathrm{GL}_2^+(\mathbb{R})$. Hence Q^* is not proper and the assertion follows.

The left translations of a normal subloop of Q^* generate a normal subgroup N of $\mathrm{GL}_2^+(\mathbb{R})$ which is the group topologically generated by all left translations of Q^*. Hence the set Λ_{Q^*} of the left translations of Q^* must contain the group $C = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \ 0 < u \in \mathbb{R} \right\}$ as a normal subgroup. The only possibility for a normal subloop of positive dimension of Q^* having $(1,0)^t$ as identity is the group $N^* := \{(s,0)^t, s \in \mathbb{R} \setminus \{0\}\}$

Lemma

If the multiplicative proper loop Q^* of a 2-dimensional locally compact connected topological quasifield Q is not quasi-simple, then the set $\mathcal{K} = \{M(r\cos(k\pi)a(r,k\pi),0); r > 0, k \in \{0,1\}\}$ of the left translations of Q^* belonging to the kernel K_r of Q has the form $\left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\}$, which is a normal subgroup of the set Λ_{Q^*} of all left translations of Q^*.

Theorem

The multiplicative loop Q^* *of a locally compact 2-dimensional quasifield* Q *with* $(1,0)^t$ *as identity of* Q^* *is proper and not quasi-simple if and only if for all* $r > 0$, $\varphi \in [0, 2\pi)$, $k \in \{0, 1\}$ *one has* $a(r, k\pi) = 1$, $b(r, k\pi) = 0$, $a(r, \varphi + k\pi) = a(1, \varphi)$ *and* $b(r, \varphi + k\pi) = b(1, \varphi)$. *Then* Q^* *is a split extension of a 1-dimensional normal subgroup* N^* *isomorphic to* $\mathbb{R} \times Z_2$, *where* Z_2 *is the group of order 2, by a subloop homeomorphic to the 1-sphere.*

Corollary

The multiplicative loop Q^* *of a locally compact 2-dimensional quasifield* Q *with* $(1,0)^t$ *as identity of* Q^* *is the direct product of the group* $\mathbb{R}$ *and a subloop homeomorphic to the 1-sphere if and only if* Q *is the field of complex numbers.*

Definition

We call the multiplicative loop Q^* of a locally compact 2-dimensional quasifield Q decomposable, *if the set of all left translations of Q^* is a product $\mathcal{T}\mathcal{K}$ with $|\mathcal{T} \cap \mathcal{K}| \leq 2$, where $\mathcal{T}$ is the set of all left translations of a 1-dimensional compact loop of the form (7) and $\mathcal{K}$ is the set of all left translations corresponding to the kernel K_r of Q.*

If the loop Q^* is decomposable, then it contains compact subloops for any $u > 0$ corresponding to the section (6). From now on we choose $u = 1$. Then the compact subloop F_u of Q^* has identity $(1, 0)^t$.
The point $(x, y)^t$ is the image of the point $(1, 0)^t$ under the linear mapping $M_{(x,y)}$ and the set $\{M_{(x,y)}; (x, y)^t \in Q^*\}$ acts sharply transitively on Q^*.

The set of the left translations of the loop Q^* is a product $\mathcal{T}\mathcal{K}$ if one has

$$\begin{pmatrix} \cos ta(1,t) & \cos tb(1,t) + \sin ta^{-1}(1,t) \\ -\sin ta(1,t) & -\sin tb(1,t) + \cos ta^{-1}(1,t) \end{pmatrix} \left[\begin{pmatrix} r\cos(k\pi)a(r,k\pi) & r\cos(k\pi)b(r,k\pi) \\ 0 & r\cos(k\pi)a^{-1}(r,k\pi) \end{pmatrix} \begin{pmatrix} u\cos\varphi a(u,\varphi) \\ -u\sin\varphi a(u,\varphi) \end{pmatrix}\right.$$

$$= \begin{pmatrix} r\cos(t+k\pi)a(r,t+k\pi) & r\cos(t+k\pi)b(r,t+k\pi) + r\sin(t+k\pi)a^{-1}(r,t+k\pi) \\ -r\sin(t+k\pi)a(r,t+k\pi) & -r\sin(t+k\pi)b(r,t+k\pi) + r\cos(t+k\pi)a^{-1}(r,t+k\pi) \end{pmatrix} \begin{pmatrix} u\cos\varphi a(u,\varphi) \\ -u\sin\varphi a(u,\varphi) \end{pmatrix}$$

The sufficient and necessary conditions for the loop Q^* to be decomposable are:

Theorem

The multiplicative loop Q^* of a locally compact connected topological 2-dimensional quasifield Q with $(1,0)^t$ as identity of Q^* is decomposable if and only if for all $r > 0$, $t \in [0, 2\pi)$, $k \in \{0,1\}$ one has

$$a(r,t+k\pi) = a(1,t)a(r,k\pi), \;\; b(r,t+k\pi) = a(1,t)b(r,k\pi) +$$

$$+a^{-1}(r,k\pi)b(1,t).$$

Theorem

If the multiplicative loop Q^* of a locally compact connected topological 2-dimensional quasifield Q with $(1,0)^t$ as identity of Q^* is not quasi-simple, then Q^* is decomposable.

If Q^* is decomposable such that its compact subloop has identity $(1,0)^t$, then $|\mathcal{T} \cap \mathcal{K}| = 2$ because one has $\mathcal{T} \cap \mathcal{K} = \left\{ I, \begin{pmatrix} -1 & -b(1,\pi) \\ 0 & -1 \end{pmatrix} \right\}$. In this case the set of all left translations of Q^* is a product $\mathcal{T}\mathcal{W}$ with $\mathcal{T} \cap \mathcal{W} = I$, where $\mathcal{W}$ is the set of all left translations corresponding to the connected component of the kernel K_r of Q.

Hence the set Λ_{Q^*} of the left translations of Q^* with a normal subloop of positive dimension and with $(1,0)^t$ as identity can be written into the form

(8)

$$\left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ua(1,t) & ub(1,t) \\ 0 & ua^{-1}(1,t) \end{pmatrix}, u > 0, t \in [0, 2\pi) \right\}$$

with $a(1, k\pi) = 1,\ b(1, k\pi) = 0,\ k \in \{0, 1\}$.

Proposition

The set Λ_{Q^*} of all left translations of the multiplicative loop Q^* for a locally compact connected topological 2-dimensional quasifield Q with $(1,0)^t$ as identity of Q^* contains the group $\mathrm{SO}_2(\mathbb{R})$ if and only if Λ_{Q^*} has the form

$$(9) \qquad \Lambda_{Q^*} = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ua(u,0) & ub(u,0) \\ 0 & ua^{-1}(u,0) \end{pmatrix} \right\}$$

$u > 0$, $t \in [0, 2\pi)$ where $a(u,0)$, $b(u,0)$ are arbitrary continuous functions with $a(u,0) > 0$ such that $ua(u,0)$ is strictly monotone. In this case Q^* is decomposable.

Corollary

If the multiplicative loop Q^* of a locally compact 2-dimensional quasifield Q has a normal subloop of positive dimension or if it contains the group $\mathrm{SO}_2(\mathbb{R})$, then Q^* is decomposable.

Proposition

The $\mathcal{C}^1$-differentiable multiplicative loop Q^* of a locally compact connected topological 2-dimensional quasifield Q with $(1,0)^t$ as identity of Q^* is decomposable precisely if for the inverse function $\bar{a}(1,t) = a^{-1}(1,t)$ and for $\bar{b}(1,t) = -b(1,t)$ the differential inequalities

$$(10) \qquad \bar{a}'^2(1,t) + \bar{b}(1,t)\bar{a}'(1,t) + \bar{b}'(1,t)\bar{a}(1,t) - \bar{a}^2(1,t) < 0,$$

$$\bar{b}'(1,0) < 1 - \bar{a}'^2(1,0)$$

are satisfied.

Corollary

Let T be any 1-dimensional $\mathcal{C}^1$-differentiable connected compact loop such that the set $\mathcal{T}$ of its left translations has the form (7) and let $\mathcal{K}$ be any set of matrices of the form

$$\mathcal{K} = \left\{ \begin{pmatrix} u\cos(k\pi)a(u,k\pi) & u\cos(k\pi)b(u,k\pi) \\ 0 & u\cos(k\pi)a^{-1}(u,k\pi) \end{pmatrix} \right\},$$

$u > 0$, $k \in \{0,1\}$ where $a(u,k\pi) > 0$ and $b(u,k\pi)$ are continuously differentiable functions such that $ua(u,0)$, $-ua(u,\pi)$ are strictly monotone. Then the product $\mathcal{T}\mathcal{K}$ is the set of all left translations of a $\mathcal{C}^1$-differentiable decomposable multiplicative loop Q^* of a 2-dimensional locally compact connected quasifield Q with $(1,0)^t$ as identity of Q^*.

D. Betten has classified all locally compact 4-dimensional translation planes which admit an at least 7-dimensional collineation group using 2-dimensional spreads. His normalized 2-dimensional spreads are sharply transitive sections $\sigma' : G/H' \to G$, where G is the connected component of the group $GL_2(\mathbb{R})$ and H' is the subgroup $\left\{ \begin{pmatrix} 1 & c \\ 0 & d \end{pmatrix}, d > 0, c \in \mathbb{R} \right\}$ (cf. D. Betten: Nicht-desarguessche 4-dimensionale Ebenen, Arch. Math.). The images of σ' has the form

(11)

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ra(r,t) & 0 \\ 0 & r^{-1}a^{-1}(r,t) \end{pmatrix} \begin{pmatrix} 1 & b(r,t)a^{-1}(r,t) \\ 0 & r^2 \end{pmatrix}.$$

With respect to the stabilizer $H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}$ the sharply transitive section σ' transforms to a sharply transitive section $\sigma : G/H \to G$ given by (1) because

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ra(r,t) & 0 \\ 0 & r^{-1}a^{-1}(r,t) \end{pmatrix} \begin{pmatrix} 1 & b(r,t)a^{-1}(r,t) \\ 0 & r^2 \end{pmatrix} =$$

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} a(r,t) & b(r,t) \\ 0 & a^{-1}(r,t) \end{pmatrix}.$$

Using Betten's classification we determined in our framework the multiplicative loops Q^* of the quasifields which coordinatize the 4-dimensional non-desarguesian translation planes $\mathcal{A}$ admitting an at least seven-dimensional collineation group and we studied their properties. Mostly we get that the loop Q^* is quasi-simple and non-decomposable. Till now mainly those simple loops have been studied for which the group generated by their left translations is a simple group. If the group generated by the left translations of a loop L is simple, then L is also simple. The multiplicative loops Q^* of 2-dimensional locally compact quasifields show that there are many interesting 2-dimensional locally compact quasi-simple loops for which the group generated by their left translations has a one-dimensional centre.

Theorem

Let $\mathcal{A}$ be a 4-dimensional locally compact non-desarguesian translation plane which admits an at least 7-dimensional collineation group Γ. Then the multiplicative loop Q^* of the quasifield Q which coordinatizes $\mathcal{A}$ is decomposable if and only if one of the following cases occurs:
(a) Γ is 8-dimensional, the translation complement C is the group $\mathrm{GL}_2(\mathbb{R})$ and acts reducibly on the translation group $\mathbb{R}^4$;
(b) Γ is 7-dimensional, the translation complement C fixes two distinct lines of $\mathcal{A}$ and leaves on one of them, one or two 1-dimensional subspaces invariant;
(c) Γ is 7-dimensional, the translation complement C fixes two distinct lines $\{S, W\}$ through the origin and acts transitively on the spaces P_S and P_W but does not act transitively on the product space $P_S \times P_W$, where P_S and P_W are the sets of all 1-dimensional subspaces of S, respectively of W.

In case a) the multiplicative loop Q_w^* is decomposable and a split extension of the normal subgroup $\widetilde{N^*} \cong \mathbb{R}$ corresponding to the connected component of $\widetilde{\mathcal{K}} = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\}$ with a subloop homeomorphic to the 1-sphere.

In case (b) the quasifields Q coordinatize either a 2-parameter family of planes or a 1-parameter family of planes. In both cases the loop Q^* is decomposable and contains the group $\mathrm{SO}_2(\mathbb{R})$.

In case (c) the quasifields $Q_{m,n,c,d}$ coordinatize a family of planes $\mathcal{A}_{m,n,c,d}$. The loops $Q^*_{m,n,c,d}$ are split extensions of the normal subgroup $\widetilde{N^*} \cong \mathbb{R}$ corresponding to the connected component of $\left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, 0 \neq u \in \mathbb{R} \right\}$ with a subloop homeomorphic to the 1-sphere.

The locally compact connected topological semifields Q such that the kernel of Q is isomorphic to $\mathbb{C}$ and Q has dimension 2 over its kernel coordinatize non-desarguesian 8-dimensional topological translation planes.
The product $x \cdot y$, $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{C}^2$ of the multiplicative loop L^* of Q is given either by

$$(12)\quad (x_1, x_2)\cdot(y_1, y_2) = \begin{pmatrix} x_1 & -e^{i\delta}\bar{x_2} \\ x_2 & \bar{x_1} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \text{ with } 0 < \delta < \pi$$

or by

$$(13)\qquad (x_1, x_2)\cdot(y_1, y_2) = \begin{pmatrix} x_1 & -c\bar{x_2} - x_2 \\ x_2 & \bar{x_1} + r\bar{x_2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where $\bar{z}$ is the complex conjugation of $z \in \mathbb{C}$, $r \geq 0$, $c \in \mathbb{C}$, $Im\ c \geq 0$ and $0 < P_{r,c}(x) = x^4 + (2\mathrm{Re}\ c - r^2)x^2 - 2rx + |c|^2 - 1$ for all $x \in \mathbb{R}$ (cf. N. Knarr: Translation planes).

The multiplicative loop L_δ given by (12) is the direct product of group $\mathbb{R}$ and a compact loop Λ_δ. We can choose as the elements of Λ_δ the matrices $l_{(x_1,x_2)} = \begin{pmatrix} x_1 & -e^{i\delta}\bar{x}_2 \\ x_2 & \bar{x}_1 \end{pmatrix}$, $|\det(l_{(x_1,x_2)})| = 1$. The product $\circ$ is given by

$$(14) \quad \begin{pmatrix} x_1 & -e^{i\delta}\bar{x}_2 \\ x_2 & \bar{x}_1 \end{pmatrix} \circ \begin{pmatrix} y_1 & -e^{i\delta}\bar{y}_2 \\ y_2 & \bar{y}_1 \end{pmatrix} = \begin{pmatrix} z_1 & -e^{i\delta}\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix},$$

where $z_1 = x_1y_1 - e^{i\delta}\bar{x}_2y_2$, $z_2 = x_2y_1 + \bar{x}_1y_2$ forms a loop $(\Lambda_\delta, \circ)$. The group of left translations as well as the group of right translations of Λ_δ is isomorphic to the group of complex 2×2-matrices their determinants have absolute value 1. The multiplication group of Λ_δ is isomorphic to $SL_4(\mathbb{R})$.

The multiplicative loop $L_{(r,c)}$ defined by (13) is the direct product of the group $\mathbb{R}$ and a loop $\Lambda_{(r,c)}$. We can choose as the elements of $\Lambda_{(r,c)}$ the matrices

$l_{(x_1,x_2)} = \begin{pmatrix} x_1 & -c\bar{x}_2 - x_2 \\ x_2 & \bar{x}_1 + r\bar{x}_2 \end{pmatrix}$, $|\det(l_{(x_1,x_2)})| = 1$. Then the product $\circ$ given by

(15)

$$\begin{pmatrix} x_1 & -c\bar{x}_2 - x_2 \\ x_2 & \bar{x}_1 + r\bar{x}_2 \end{pmatrix} \circ \begin{pmatrix} y_1 & -c\bar{y}_2 - y_2 \\ y_2 & \bar{y}_1 + r\bar{y}_2 \end{pmatrix} = \begin{pmatrix} z_1 & -c\bar{z}_2 - z_2 \\ z_2 & \bar{z}_1 + r\bar{z}_2 \end{pmatrix},$$

where $z_1 = x_1 y_1 - c\bar{x}_2 y_2 - x_2 y_2$, $z_2 = x_2 y_1 + \bar{x}_1 y_2 + r\bar{x}_2 y_2$ forms a loop $(\Lambda_{(r,c)}, \circ)$ diffeomorphic to S^3. The group of left translations of the loop $\Lambda_{(r,c)}$ is the group of complex 2×2-matrices their determinants have absolute value 1. The group of right translations as well as the multiplication group of $\Lambda_{(r,c)}$ are isomorphic to $SL_4(\mathbb{R})$.

The translation planes $\mathcal{A}$ have been a popular subject of geometrical research. The translation group of $\mathcal{A}$ is isomorphic to the additive group of a vector space V over a skewfield. Every translation plane can be obtained from a spread of a suitable vector space V.

Definition

Let V be a vector space over a skewfield. A collection $\mathcal{B}$ of subspaces of V with $|\mathcal{B}| \geq 3$ is called a spread of V if for any two different elements U_1, $U_2 \in \mathcal{B}$ we have $V = U_1 \oplus U_2$ and every vector of V is contained in an element of $\mathcal{B}$.

The points of $\mathcal{A}$ can be identified with the elements of V. The lines through the origin form a spread $\mathcal{B}$ of V, the other lines are cosets of the elements of $\mathcal{B}$.

They coordinatize locally compact 4-dimensional topological translation planes. The classification of these topological translation planes $\mathcal{A}$ was accomplished by reconstructing the spreads corresponding to $\mathcal{A}$ from the translation complement which is the stabilizer of a point in the collineation group of $\mathcal{A}$. In this way all planes $\mathcal{A}$ having an at least 7-dimensional collineation group have been determined by D. Betten.

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