

# Multiplicative loops of 2-dimensional topological quasifields

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## Abstract

We determine the algebraic structure of the multiplicative loops for locally compact 2-dimensional topological connected quasifields. In particular, our attention turns to multiplicative loops which have either a normal subloop of positive dimension or which contain a 1-dimensional compact subgroup. In the last section we determine explicitly the quasifields which coordinatize locally compact translation planes of dimension 4 admitting an at least 7-dimensional Lie group as collineation group.

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## 1. Introduction

Locally compact connected topological non-desarguesian translation planes have been a popular subject of geometrical research since the seventies of the last century ([18], [2]-[9], [13], [15]). These planes are coordinatized by locally compact quasifields  $Q$  such that the kernel of  $Q$  is either the field  $\mathbb{R}$  of real numbers or the field  $\mathbb{C}$  of complex numbers (cf. [11], IX.5.5 Theorem, p. 323). If the quasifield  $Q$  is 2-dimensional, then its kernel is  $\mathbb{R}$ .

The classification of topological translation planes  $\mathcal{A}$  was accomplished by reconstructing the spreads corresponding to  $\mathcal{A}$  from the translation complement which is the stabilizer of a point in the collineation group of  $\mathcal{A}$ . In this way all planes  $\mathcal{A}$  having an at least 7-dimensional collineation group have been determined ([3]-[8], [15]).

Although any spread gives the lines through the origin and hence the multiplication in a 2-dimensional quasifield  $Q$  coordinatizing the plane  $\mathcal{A}$ , to the algebraic structure of the multiplicative loop  $Q^*$  of a proper quasifield  $Q$

is not given special attention apart from the facts that the group topologically generated by the left translations of  $Q^*$  is the connected component of  $GL_2(\mathbb{R})$ , the group topologically generated by the right translations of  $Q^*$  is an infinite-dimensional Lie group (cf. [14], Section 29, p. 345) and any locally compact 2-dimensional semifield is the field of complex numbers ([17]).

Since in the meantime some progress in the classification of compact differentiable loops on the 1-sphere has been achieved (cf. [10]), we believe that loops could have more space in the research concerning 4-dimensional translation planes. Using the images of differentiable sections  $\sigma : G/H \rightarrow G$ , where  $H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}$ , we classify the  $C^1$ -differentiable multiplicative loops  $Q^*$  of 2-dimensional locally compact quasifields  $Q$  by functions, the Fourier series of which are described in [10].

The multiplicative loops  $Q^*$  of 2-dimensional locally compact left quasifields  $Q$  for which the set of the left translations of  $Q^*$  is the product  $\mathcal{TK}$  with  $|\mathcal{T} \cap \mathcal{K}| \leq 2$ , where  $\mathcal{T}$  is the set of the left translations of a 1-dimensional compact loop and  $\mathcal{K}$  is the set of the left translations of  $Q^*$  corresponding to the kernel  $K_r$  of  $Q$ , form an important subclass of loops, that we call decomposable loops. Namely, if  $Q^*$  has a normal subloop of positive dimension or if it contains the group  $SO_2(\mathbb{R})$ , then  $Q^*$  is decomposable. Moreover, we show that any 1-dimensional  $C^1$ -differentiable compact loop is a factor of a decomposable multiplicative loop of a locally compact connected quasifield coordinatizing a 4-dimensional translation plane. A 2-dimensional locally compact quasifield  $Q$  is the field of complex numbers if and only if the multiplicative loop  $Q^*$  contains a 1-dimensional normal compact subloop.

Till now mainly those simple loops have been studied for which the group generated by their left translations is a simple group. If the group generated by the left translations of a loop  $L$  is simple, then  $L$  is also simple (cf. Lemma 1.7 in [14]). The multiplicative loops  $Q^*$  of 2-dimensional locally compact quasifields show that there are many interesting 2-dimensional locally compact quasi-simple loops for which the group generated by their left translations has a one-dimensional centre.

In the last section we use Betten's classification to determine in our framework the multiplicative loops  $Q^*$  of the quasifields which coordinatize the 4-dimensional non-desarguesian translation planes  $\mathcal{A}$  admitting an at least seven-dimensional collineation group and to study their properties. The results obtained there yield the following

**Theorem** *Let  $\mathcal{A}$  be a 4-dimensional locally compact non-desarguesian translation plane which admits an at least 7-dimensional collineation group  $\Gamma$ . If the quasifield  $Q$  coordinatizing  $\mathcal{A}$  is constructed with respect to two lines such that their intersection points with the line at infinity are contained in the*

1-dimensional orbit of  $\Gamma$  or contain the set of the fixed points of  $\Gamma$ , then the multiplicative loop  $Q^*$  of  $Q$  is decomposable if and only if one of the following cases occurs:

- (a)  $\Gamma$  is 8-dimensional, the translation complement  $C$  is the group  $\text{GL}_2(\mathbb{R})$  and acts reducibly on the translation group  $\mathbb{R}^4$ ;
- (b)  $\Gamma$  is 7-dimensional, the translation complement  $C$  fixes two distinct lines of  $\mathcal{A}$  and leaves on one of them, one or two 1-dimensional subspaces invariant;
- (c)  $\Gamma$  is 7-dimensional, the translation complement  $C$  fixes two distinct lines  $\{S, W\}$  through the origin and acts transitively on the spaces  $P_S$  and  $P_W$  but does not act transitively on the product space  $P_S \times P_W$ , where  $P_S$  and  $P_W$  are the sets of all 1-dimensional subspaces of  $S$ , respectively of  $W$ .

## 2. Preliminaries

A binary system  $(L, \cdot)$  is called a quasigroup if for any given  $a, b \in L$  the equations  $a \cdot y = b$  and  $x \cdot a = b$  have unique solutions which we denote by  $y = a \setminus b$  and  $x = b / a$ . If a quasigroup  $L$  has an element 1 such that  $x = 1 \cdot x = x \cdot 1$  holds for all  $x \in L$ , then it is called a loop and 1 is the identity element of  $L$ . The left translations  $\lambda_a : L \rightarrow L, x \mapsto a \cdot x$  and the right translations  $\rho_a : L \rightarrow L, x \mapsto x \cdot a, a \in L$ , are bijections of  $L$ . Two loops  $(L_1, \circ)$  and  $(L_2, *)$  are called isotopic if there exist three bijections  $\alpha, \beta, \gamma : L_1 \rightarrow L_2$  such that  $\alpha(x) * \beta(y) = \gamma(x \circ y)$  holds for all  $x, y \in L_1$ . A binary system  $(K, \cdot)$  is called a subloop of  $(L, \cdot)$  if  $K \subset L$ , for any given  $a, b \in K$  the equations  $a \cdot y = b$  and  $x \cdot a = b$  have unique solutions in  $K$  and  $1 \in K$ . The kernel of a homomorphism  $\alpha : (L, \cdot) \rightarrow (L', *)$  of a loop  $L$  into a loop  $L'$  is a normal subloop  $N$  of  $L$ , i.e. a subloop of  $L$  such that

$$x \cdot N = N \cdot x, (x \cdot N) \cdot y = x \cdot (N \cdot y), (N \cdot x) \cdot y = N \cdot (x \cdot y) \quad (1)$$

hold for all  $x, y \in L$ . A loop  $L$  is called simple if  $\{1\}$  and  $L$  are its only normal subloops.

A loop  $L$  is called topological, if it is a topological space and the binary operations  $(a, b) \mapsto a \cdot b, (a, b) \mapsto a \setminus b, (a, b) \mapsto b / a : L \times L \rightarrow L$  are continuous. Then the left and right translations of  $L$  are homeomorphisms of  $L$ . If  $L$  is a connected differentiable manifold such that the loop multiplication and the left division are continuously differentiable mappings, then we call  $L$  an almost  $\mathcal{C}^1$ -differentiable loop. If also the right division of  $L$  is continuously differentiable, then  $L$  is a  $\mathcal{C}^1$ -differentiable loop. A connected topological loop is quasi-simple if it contains no normal subloop of positive dimension. Every topological, respectively almost  $\mathcal{C}^1$ -differentiable, connected loop  $L$  having a Lie group  $G$  as the group topologically generated by the left trans-

lations of  $L$  corresponds to a sharply transitive continuous, respectively  $\mathcal{C}^1$ -differentiable section  $\sigma : G/H \rightarrow G$ , where  $G/H = \{xH | x \in G\}$  consists of the left cosets of the stabilizer  $H$  of  $1 \in L$  such that  $\sigma(H) = 1_G$  and  $\sigma(G/H)$  generates  $G$ . The section  $\sigma$  is sharply transitive if the image  $\sigma(G/H)$  acts sharply transitively on the factor space  $G/H$ , i.e. for given left cosets  $xH, yH$  there exists precisely one  $z \in \sigma(G/H)$  which satisfies the equation  $zxH = yH$ .

A (left) quasifield is an algebraic structure  $(Q, +, \cdot)$  such that  $(Q, +)$  is an abelian group with neutral element 0,  $(Q \setminus \{0\}, \cdot)$  is a loop with identity element 1 and between these operations the (left) distributive law  $x \cdot (y + z) = x \cdot y + x \cdot z$  holds. A locally compact connected topological quasifield is a locally compact connected topological space  $Q$  such that  $(Q, +)$  is a topological group,  $(Q \setminus \{0\}, \cdot)$  is a topological loop, the multiplication  $\cdot : Q \times Q \rightarrow Q$  is continuous and the mappings  $\lambda_a : x \mapsto a \cdot x$  and  $\rho_a : x \mapsto x \cdot a$  with  $0 \neq a \in Q$  are homeomorphisms of  $Q$ . If for any given  $a, b, c \in Q$  the equation  $x \cdot a + x \cdot b = c$  with  $a + b \neq 0$  has precisely one solution, then  $Q$  is called planar. A translation plane is an affine plane with transitive group of translations; this is coordinatized by a planar quasifield (cf. [16], Kap. 8).

The kernel  $K_r$  of a (left) quasifield  $Q$  is a skewfield defined by

$$K_r = \{k \in Q; (x+y) \cdot k = x \cdot k + y \cdot k \text{ and } (x \cdot y) \cdot k = x \cdot (y \cdot k) \text{ for all } x, y \in Q\}.$$

In this paper we consider left quasifields  $Q$ . Then  $Q$  is a right vector space over  $K_r$ . Moreover, for all  $a \in Q$  the map  $\lambda_a : Q \rightarrow Q, x \mapsto a \cdot x$  is  $K_r$ -linear. According to [12], Theorem 7.3, p. 160, every quasifield that has finite dimension over its kernel is planar.

Let  $F$  be a skewfield and let  $V$  be a vector space over  $F$ . A collection  $\mathcal{B}$  of subspaces of  $V$  with  $|\mathcal{B}| \geq 3$  is called a spread of  $V$  if for any two different elements  $U_1, U_2 \in \mathcal{B}$  we have  $V = U_1 \oplus U_2$  and every vector of  $V$  is contained in an element of  $\mathcal{B}$ .

If  $S$  and  $W$  are different subspaces of the spread  $\mathcal{B}$ , then  $V$  can be coordinatized in such a way that  $S = \{0\} \times X$  and  $W = X \times \{0\}$ . Any spread of  $V = X \times X$  can be described by a collection  $\mathcal{M}$  of linear mappings  $X \rightarrow X$  satisfying the following conditions:

- ( $M_1$ ) For any  $\omega_1 \neq \omega_2 \in \mathcal{M}$  the mapping  $\omega_1 - \omega_2$  is bijective.
  - ( $M_2$ ) For all  $x \in X \setminus \{0\}$  the mapping  $\phi_x : \mathcal{M} \rightarrow X : \omega \mapsto \omega(x)$  is surjective.
- Namely, if  $\mathcal{M}$  is a collection of linear mappings satisfying ( $M_1$ ) and ( $M_2$ ), then the sets  $U_\omega = \{(x, \omega(x)), x \in X\}$  and  $\{0\} \times X$  yield a spread of  $V = X \times X$ . Conversely, every component  $U \in \mathcal{B} \setminus \{S\}$  of  $V$  is the graph of a linear mapping  $\omega_U : W \rightarrow S$  and the set of  $\omega_U$  gives a collection  $\mathcal{M}$  of linear mappings of  $X$  satisfying ( $M_1$ ) and ( $M_2$ ) (cf. [13], Proposition 1.11.).

The mapping  $\omega_W$  is the zero mapping. For this reason any collection  $\mathcal{M}$  of linear mappings of  $X$  which satisfy  $(M_1)$  and  $M_2$  is called a spread set of  $X$ . Every translation plane can be obtained from a spread set of a suitable vector space  $V = X \times X$  (cf. [13], Theorem 1.5, p. 7, and [1]). As every translation plane can be coordinatized by a quasifield and a quasifield contains 0 and 1, the associated spread set contains the zero endomorphism and the identity map. This is not true for arbitrary spread sets  $\mathcal{M}$ , but if  $\omega_0, \omega_1 \in \mathcal{M}$  are distinct, then  $\mathcal{M}' = \{(\omega - \omega_0)(\omega_1 - \omega_0)^{-1}, \omega \in \mathcal{M}\}$  is a normalized spread set of  $X$ , i.e. a spread set which contains the zero and the identity map. The translation planes obtained from  $\mathcal{M}$  and  $\mathcal{M}'$  are isomorphic (cf. [13], Lemma 1.15, p. 13). Let  $\mathcal{M}$  be a normalized spread set of  $X$ ,  $e \in X \setminus \{0\}$  and let  $\phi_e : \mathcal{M} \rightarrow X$  be defined by  $\phi_e(\omega) = \omega(e)$ . Then the multiplication  $\circ : X \times X \rightarrow X$  defined by  $m \circ x = (\phi_e^{-1}(m))(x)$  yields a multiplicative loop of a left quasifield  $Q$  coordinatizing the translation plane  $\mathcal{A}$  belonging to the spread  $\mathcal{M}$  of  $X$ .

If we fix a basis of  $Q$  over its kernel  $K_r$  and identify  $X$  with the vector space of pairs  $\{(x, y)^t, x, y \in K_r\}$ , then the set  $\mathcal{M}$  consists of matrices  $C(\alpha, \beta, \gamma, \delta) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \alpha, \beta, \gamma, \delta \in K_r$ . If  $e = (1, 0)^t$ , then we get  $\phi_e(C(\alpha, \beta, \gamma, \delta)) = C(\alpha, \beta, \gamma, \delta)(e) = (\alpha, \gamma)^t$ . Since  $\mathcal{M}$  is a spread of  $X$  the set of vectors  $(\alpha, \gamma)^t$  consists of all vectors of  $X$ . Hence if  $(\alpha, \gamma)^t$  is an element of  $X$ , then there exists a unique matrix of  $\mathcal{M}$  having  $(\alpha, \gamma)^t$  as the first column.

We consider multiplicative loops of locally compact connected topological quasifields  $Q$  of dimension 2 coordinatizing 4-dimensional non-desarguesian topological translation planes. Then the kernel  $K_r$  of  $Q$  is isomorphic to the field of the real numbers,  $(Q, +)$  is the vector group  $\mathbb{R}^2$  and the multiplicative loop  $(Q \setminus \{0\}, \cdot)$  is homeomorphic to  $\mathbb{R} \times S^1$ , where  $S^1$  is the circle.

#### 4. Multiplicative loops of 2-dimensional quasifields

Let  $(Q, +, *)$  be a real topological (left) quasifield of dimension 2. Let  $e_1$  be the identity element of the multiplicative loop  $Q^* = (Q \setminus \{0\}, *)$  of  $Q$ , which generates the kernel  $K_r = \mathbb{R}$  of  $Q$  as a vector space and let  $B = \{e_1, e_2\}$  be a basis of the right vector space  $Q$  over  $K_r$ . Once we fix  $B$ , we identify  $Q$  with the vector space of pairs  $(x, y)^t \in \mathbb{R}^2$  and  $K_r$  with the subspace of pairs  $(x, 0)^t$ . The element  $(1, 0)^t$  is the identity element of  $Q^*$ . According to [14], Theorem 29.1, p. 345, the group  $G$  topologically generated by the left translations of  $Q^*$  is the connected component of the group  $\text{GL}_2(\mathbb{R})$ . As  $\dim Q^* = 2$  and the stabilizer of the identity element of  $Q^*$  in  $G$  does not contain any non-trivial normal subgroup of  $G$  we may replace the stabilizer of the identity by the subgroup  $H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}$ . The elements  $g$  of  $G$  have a

unique decomposition as the product

$$g = \begin{pmatrix} u \cos t & u \sin t \\ -u \sin t & u \cos t \end{pmatrix} \begin{pmatrix} k & l \\ 0 & k^{-1} \end{pmatrix}$$

with suitable elements  $u > 0$ ,  $k > 0$ ,  $l \in \mathbb{R}$ ,  $t \in [0, 2\pi)$ . Hence the loop  $Q^*$  corresponds to a continuous section  $\sigma : G/H \rightarrow G$ ;

$$\begin{pmatrix} u \cos t & u \sin t \\ -u \sin t & u \cos t \end{pmatrix} H \mapsto \begin{pmatrix} u \cos t & u \sin t \\ -u \sin t & u \cos t \end{pmatrix} \begin{pmatrix} a(u, t) & b(u, t) \\ 0 & a^{-1}(u, t) \end{pmatrix} \quad (2)$$

where the pair of continuous functions  $a(u, t), b(u, t) : \mathbb{R}_{>0} \times [0, 2\pi) \rightarrow \mathbb{R}$ , where  $\mathbb{R}_{>0}$  is the set of positive numbers, satisfies the following conditions:

$$a(u, t) > 0, \quad a(1, 0) = 1, \quad b(1, 0) = 0.$$

As  $Q$  is a left quasifield, any  $(x, y)^t \in Q^*$  induces a linear transformation  $M_{(x,y)} \in \sigma(G/H)$ . More precisely one has

$$\begin{pmatrix} x \\ y \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix} = M_{(x,y)} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} r \cos \varphi & r \sin \varphi \\ -r \sin \varphi & r \cos \varphi \end{pmatrix} \begin{pmatrix} a(r, \varphi) & b(r, \varphi) \\ 0 & a^{-1}(r, \varphi) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad (3)$$

where  $x = r \cos(\varphi)a(r, \varphi)$ ,  $y = -r \sin(\varphi)a(r, \varphi)$ . The kernel  $K_r$  of  $Q$  consists of  $(0, 0)^t$  and  $(r \cos(k\pi)a(r, k\pi), 0)^t$ ,  $r > 0$ ,  $k \in \{0, 1\}$ , such that the matrices corresponding to the elements  $(r \cos(k\pi)a(r, k\pi), 0)^t$  have the form

$$M(r \cos(k\pi)a(r, k\pi), 0) = \begin{pmatrix} r \cos(k\pi)a(r, k\pi) & r \cos(k\pi)b(r, k\pi) \\ 0 & r \cos(k\pi)a^{-1}(r, k\pi) \end{pmatrix}.$$

The identity matrix  $I$  corresponds to the identity  $(1, 0)^t$  of  $Q^*$ . As to each real number  $r \cos(k\pi)a(r, k\pi)$  belongs precisely one matrix

$$M(r \cos(k\pi)a(r, k\pi), 0)$$

the functions  $f_1(r) = ra(r, 0)$ ,  $r > 0$  and  $f_2(r) = -ra(r, \pi)$ ,  $r > 0$  are strictly monotone. If the functions  $a(r, 0)$ ,  $a(r, \pi)$  are differentiable, then for every  $r > 0$  the derivatives  $a(r, 0) + ra'(r, 0)$  and  $-a(r, \pi) - ra'(r, \pi)$  are either always positive or negative. This is equivalent to the fact that the derivatives  $[\ln(a(r, 0))]', [\ln(a(r, \pi))]'$  are always greater or smaller than  $-r^{-1}$ . Since the matrix  $M(r \cos(k\pi)a(r, k\pi), 0)$  is isomorphic to the group  $(\mathbb{R}, +)$  the function  $a(r, 0)$  is a homomorphism and for the function  $b(r, 0)$  the identity  $b(r_1 r_2, 0) = a(r_1, 0)b(r_2, 0) + b(r_1, 0)a^{-1}(r_2, 0)$  is satisfied for all  $r_1, r_2 > 0$ .

**Remark 1.** The set  $\mathcal{K} = \{M(r \cos(k\pi)a(r, k\pi), 0); r > 0, k \in \{0, 1\}\}$  of the left translations of  $Q^*$  corresponding to the kernel  $K_r$  of  $Q$  is

$$\left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, r \in \mathbb{R} \setminus \{0\} \right\}$$

if and only if one has  $a(r, k\pi) = 1$ ,  $b(r, k\pi) = 0$  for all  $r > 0$ ,  $k \in \{0, 1\}$ .

The section  $\sigma$  given by (2) is sharply transitive precisely if for all pairs  $(u_1, t_1), (u_2, t_2)$  in  $\mathbb{R}_{>0} \times [0, 2\pi)$  there exists precisely one  $(u, t) \in \mathbb{R}_{>0} \times [0, 2\pi)$  and  $k > 0, l \in \mathbb{R}$  such that

$$\begin{pmatrix} u \cos t & u \sin t \\ -u \sin t & u \cos t \end{pmatrix} \begin{pmatrix} a(u, t) & b(u, t) \\ 0 & a^{-1}(u, t) \end{pmatrix} \begin{pmatrix} u_1 \cos t_1 & u_1 \sin t_1 \\ -u_1 \sin t_1 & u_1 \cos t_1 \end{pmatrix} = \\ \begin{pmatrix} u_2 \cos t_2 & u_2 \sin t_2 \\ -u_2 \sin t_2 & u_2 \cos t_2 \end{pmatrix} \begin{pmatrix} k & l \\ 0 & k^{-1} \end{pmatrix}. \quad (4)$$

As the determinant of the matrices on both sides of (4) are equal we get that  $u = u_1^{-1}u_2$ . Therefore the system (4) of equations is uniquely solvable if and only if for any fixed  $u > 0$  the mapping

$$\sigma_u : \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} H \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(u, t) & b(u, t) \\ 0 & a^{-1}(u, t) \end{pmatrix}$$

determines a quasigroup  $F_u$  homeomorphic to  $S^1$ . One may take as the points of  $F_u$  the vectors  $(ua(u, t)a^{-1}(u, 0) \cos t, -ua(u, t)a^{-1}(u, 0) \sin t)^t$  and as the section the mapping

$$\sigma_u : \begin{pmatrix} ua(u, t)a^{-1}(u, 0) \cos t \\ -ua(u, t)a^{-1}(u, 0) \sin t \end{pmatrix} \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(u, t)a^{-1}(u, 0) & b(u, t) \\ 0 & a^{-1}(u, t)a(u, 0) \end{pmatrix} = \\ \begin{pmatrix} a(u, t)a^{-1}(u, 0) \cos t & b(u, t) \cos t + a^{-1}(u, t)a(u, 0) \sin t \\ -a(u, t)a^{-1}(u, 0) \sin t & -b(u, t) \sin t + a^{-1}(u, t)a(u, 0) \cos t \end{pmatrix}. \quad (5)$$

In this way we see that the quasigroup  $F_u$  has the right identity  $(u, 0)^t$  since

$$\sigma_u \begin{pmatrix} ua(u, t)a^{-1}(u, 0) \cos t \\ -ua(u, t)a^{-1}(u, 0) \sin t \end{pmatrix} \cdot \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} ua(u, t)a^{-1}(u, 0) \cos t \\ -ua(u, t)a^{-1}(u, 0) \sin t \end{pmatrix}.$$

The quasigroup  $F_u$  is a loop, i.e.  $(u, 0)^t$  is the left identity of  $F_u$ , if and only if

$$\sigma_u \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} a(u, 0)a^{-1}(u, 0) \cos 0 & b(u, 0) \cos 0 \\ 0 & a^{-1}(u, 0)a(u, 0) \cos 0 \end{pmatrix} = \begin{pmatrix} 1 & b(u, 0) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which means  $b(u, 0) = 0$  for all  $u > 0$ . The almost  $\mathcal{C}^1$ -differentiable loop  $Q^*$  belonging to the sharply transitive  $\mathcal{C}^1$ -differentiable section  $\sigma$  given by (2) is  $\mathcal{C}^1$ -differentiable precisely if the mapping  $(xH, yH) \mapsto z : G/H \times G/H \rightarrow \sigma(G/H)$  determined by  $zxH = yH$  is  $\mathcal{C}^1$ -differentiable (cf. [14], p. 32), i.e. the solutions  $u > 0, t \in [0, 2\pi)$  of the matrix equation (4) are continuously differentiable functions of  $u_1, u_2 \in \mathbb{R}_{>0}, t_1, t_2 \in [0, 2\pi)$ . The function  $u = u_1^{-1}u_2$  is continuously differentiable. If for each fixed  $u > 0$  the section  $\sigma_u$  given by (5) yields a 1-dimensional  $\mathcal{C}^1$ -differentiable compact loop, then the function  $t(u_1, u_2, t_1, t_2) = t_{(u_1, u_2)}(t_1, t_2)$  is continuously differentiable (cf. [14], Examples 20.3, p. 258). Indeed, the function  $t_{(u_1, u_2)}(t_1, t_2)$  is determined implicitly by equations which depend continuously differentiably also on the parameters  $u_1$  and  $u_2$ . Applying the above discussion we can prove the following:

**Theorem 2.** *Let  $Q^*$  be the  $\mathcal{C}^1$ -differentiable multiplicative loop of a locally compact 2-dimensional connected topological quasifield  $Q$ . Then  $Q^*$  is diffeomorphic to  $S^1 \times \mathbb{R}$  and belongs to a  $\mathcal{C}^1$ -differentiable sharply transitive section  $\sigma$  of the form*

$$\begin{pmatrix} u \cos t & u \sin t \\ -u \sin t & u \cos t \end{pmatrix} H \mapsto \begin{pmatrix} u \cos t & u \sin t \\ -u \sin t & u \cos t \end{pmatrix} \cdot \begin{pmatrix} a(u, t) & b(u, t) \\ 0 & a^{-1}(u, t) \end{pmatrix},$$

with  $b(u, 0) = 0$  for all  $u > 0$  if and only if for each fixed  $u > 0$  the function  $a_u^{-1}(t) := a(u, 0)a^{-1}(u, t)$  has the shape

$$a_u^{-1}(t) = e^t \left( 1 - \int_0^t R(s) e^{-s} ds \right)$$

where  $R(s)$  is a continuous function, the Fourier series of which is contained in the set  $\mathcal{F}$  of Definition 1 in [10] and converges uniformly to  $R$ . Moreover,  $b_u(t) := b(u, t)$  is a periodic  $\mathcal{C}^1$ -differentiable function with  $b_u(0) = b_u(2\pi) = 0$  such that

$$b_u(t) > -a_u(t) \int_0^t \frac{(a_u^2(s) - a_u'^2(s))}{a_u^4(s)} ds \quad \text{for all } t \in (0, 2\pi).$$

*Proof.* The section  $\sigma_u$  given by (5) yields a 1-dimensional  $\mathcal{C}^1$ -differentiable compact loop having the group  $\text{SL}_2(\mathbb{R})$  as the group topologically generated by its left translations if and only if for each fixed  $u > 0$  the continuously differentiable functions  $a(u, 0)a^{-1}(u, t) := \bar{a}_u(t)$ ,  $-b(u, t) := \bar{b}_u(t)$  satisfy the conditions

$$\bar{a}_u'^2(t) + \bar{b}_u(t)\bar{a}_u'(t) + \bar{b}_u'(t)\bar{a}_u(t) - \bar{a}_u^2(t) < 0, \quad \bar{b}_u'(0) < 1 - \bar{a}_u'^2(0) \quad (6)$$

(cf. [14], Section 18, (C), p. 238, [10], pp. 132-139). The solution of the differential inequalities (6) is given by Theorem 6 in [10], pp. 138-139. This proves the assertion.  $\square$

**Proposition 3.** *Let  $Q^*$  be the  $\mathcal{C}^1$ -differentiable multiplicative loop of a locally compact 2-dimensional connected topological quasifield  $Q$ . Assume that for each fixed  $u > 0$  the function  $a_u(t) := a^{-1}(u, 0)a(u, t)$  is the constant function 1 and that  $b(u, 0) = 0$  is satisfied for all  $u > 0$ . Then  $Q^*$  belongs to a  $\mathcal{C}^1$ -differentiable sharply transitive section  $\sigma$  of the form (2) if and only if for each fixed  $u > 0$  one has  $b_u(t) := b(u, t) > -t$  for all  $0 < t < 2\pi$ .*



*Proof.* If for each fixed  $u > 0$  the function  $a(u, 0)a^{-1}(u, t) = a_u^{-1}(t) = \bar{a}_u(t)$  is constant with value 1, then the section  $\sigma_u$  given by (5) yields a  $\mathcal{C}^1$ -differentiable compact loop  $L$  if and only if for each fixed  $u > 0$  the continuously differentiable function  $\bar{b}_u(t) := -b_u(t)$  satisfies the differential inequality  $\bar{b}'_u(t) < 1$  with the initial condition  $\bar{b}'_u(0) < 1$  (cf. (6)). This is the case precisely if one has  $b_u(t) > -t$  for all  $0 < t < 2\pi$ .  $\square$

**Proposition 4.** *Let  $Q^*$  be the  $\mathcal{C}^1$ -differentiable multiplicative loop of a locally compact 2-dimensional connected topological quasifield  $Q$ . Assume that for each fixed  $u > 0$  the function  $b(u, t)$  is the constant function 0. Then  $Q^*$  belongs to a  $\mathcal{C}^1$ -differentiable sharply transitive section  $\sigma$  of the form (2) precisely if for each fixed  $u > 0$  one has  $e^{-t} < a(u, t)a^{-1}(u, 0) < e^t$  for all  $0 < t < 2\pi$ .*

*Proof.* If for each fixed  $u > 0$  the function  $b(u, t) = -\bar{b}_u(t)$  is constant with value 0, then the section  $\sigma_u$  given by (5) determines a  $\mathcal{C}^1$ -differentiable compact loop  $L$  if and only if for each fixed  $u > 0$  the following inequalities are satisfied:

$$(\bar{a}'_u(t) - \bar{a}_u(t))(\bar{a}'_u(t) + \bar{a}_u(t)) < 0, \quad 0 < 1 - \bar{a}_u^2(0),$$

where  $\bar{a}_u(t) = a(u, 0)a^{-1}(u, t)$ . This is the case precisely if either one has  $\bar{a}'_u(t) - \bar{a}_u(t) < 0$  and  $\bar{a}'_u(t) + \bar{a}_u(t) > 0$  or one has  $\bar{a}'_u(t) - \bar{a}_u(t) > 0$  and  $\bar{a}'_u(t) + \bar{a}_u(t) < 0$ . Now we consider the first case. Then the function  $\bar{a}_u(t)$  determines a loop if and only if for each fixed  $u > 0$  it is a subfunction of a differentiable function  $h_u(t) := h(u, t)$  with  $h_u(0) = 1$ ,  $h_u^2(0) = 1$ ,  $h'_u(t) = h_u(t)$  and an upper function of a differentiable function  $l_u(t) := l(u, t)$  with  $l_u(0) = 1$ ,  $l_u^2(0) = 1$ ,  $l'_u(t) = -l_u(t)$  (cf. [19], p. 66). Hence for each fixed  $u > 0$  the function  $\bar{a}_u(t)$  is a subfunction of the function  $e^t$  and an upper function of the function  $e^{-t}$  for all  $t \in (0, 2\pi)$ . Therefore, any continuously differentiable function  $\bar{a}_u(t)$  such that for each fixed  $u > 0$  and for all  $t \in (0, 2\pi)$  one has  $e^{-t} < \bar{a}_u(t)^{-1} < e^t$  determines a  $\mathcal{C}^1$ -differentiable compact loop  $L$ .

In the second case an analogous consideration as in the first case gives that for all fixed  $u > 0$  the function  $a(u, t)a^{-1}(u, 0)$  must be a subfunction of the function  $e^{-t}$  and an upper function of the function  $e^t$  for all  $t \in (0, 2\pi)$ . Hence in this case the function  $a(u, t)a^{-1}(u, 0)$  does not exist.  $\square$

**Proposition 5.** *Let*

$$\begin{pmatrix} u \cos t & u \sin t \\ -u \sin t & u \cos t \end{pmatrix} H \mapsto \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1, t) & b(u, t) \\ 0 & a^{-1}(1, t) \end{pmatrix}, \quad u > 0, t \in [0, 2\pi) \quad (7)$$

*with  $b(u, 0) = 0$  for all  $u > 0$  be a section belonging to a multiplicative loop  $Q^*$  of a locally compact 2-dimensional connected topological quasifield  $Q$ . Then  $Q^*$  contains for any  $u > 0$  a 1-dimensional compact subloop.*

*Proof.* The image of the section (7) acts sharply transitively on the point set  $\mathbb{R}^2 \setminus \{(0,0)^t\}$ . Since the subgroup  $\left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, u > 0 \right\}$  leaves any line through  $(0,0)^t$  fixed, the subset

$$\mathcal{T} = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1,t) & b(u,t) \\ 0 & a^{-1}(1,t) \end{pmatrix}, t \in [0, 2\pi) \right\} \quad (8)$$

acts sharply transitively on the oriented lines through  $(0,0)^t$  for any  $u > 0$ . Therefore  $\mathcal{T}$  corresponds to a 1-dimensional compact loop since  $b(u,0) = 0$  for all  $u > 0$ .  $\square$

As  $\mathcal{T}$  given by (8) is the image of a section corresponding to a 1-dimensional compact subloop of  $Q^*$ , every element of  $\mathcal{T}$  is elliptic.

**Proposition 6.** *Every element of the set  $\mathcal{T}$  given by (8) is elliptic if and only if the following holds:  $a(1, k\pi) = 1$  for  $k \in \{0, 1\}$ :*

1) *if for all  $t \in [0, 2\pi)$  and  $u > 0$  one has  $b(u, t) = 0$ , then the function  $a(1, t)$  satisfies the inequalities:*

$$\frac{1 - |\sin(t)|}{|\cos(t)|} \leq a(1, t) \leq \frac{1 + |\sin(t)|}{|\cos(t)|}, \quad (9)$$

2) *if the function  $b(u, t)$  is different from the constant function 0, then for  $\sin(t) > 0$  one has*

$$\frac{(a(1, t) + a(1, t)^{-1}) \cos(t) - 2}{\sin(t)} < b(u, t) < \frac{(a(1, t) + a(1, t)^{-1}) \cos(t) + 2}{\sin(t)}, \quad (10)$$

for  $\sin(t) < 0$  we have

$$\frac{(a(1, t) + a(1, t)^{-1}) \cos(t) + 2}{\sin(t)} < b(u, t) < \frac{(a(1, t) + a(1, t)^{-1}) \cos(t) - 2}{\sin(t)}. \quad (11)$$

*Proof.* Any element of (8) is elliptic if and only if the inequality

$$|\cos(t)(a(1, t) + a(1, t)^{-1}) - \sin(t)b(u, t)| \leq 2 \quad (12)$$

holds, where the equality sign occurs only for  $t = k\pi$ ,  $k \in \{0, 1\}$ . Hence  $a(1, k\pi) = 1$ . If  $b(u, t) = 0$ , then inequality (12) reduces to  $a^2(1, t)|\cos(t)| - 2a(1, t) + |\cos(t)| \leq 0$  which is equivalent to inequalities (9). If  $b(u, t) \neq 0$ , then inequality (12) is equivalent for all  $t \neq k\pi$ ,  $k \in \{0, 1\}$ , to

$$(a(1, t) + a(1, t)^{-1})^2 \cos^2(t) - 2(a(1, t) + a(1, t)^{-1}) \sin(t) \cos(t) b(u, t) + \sin^2(t) b^2(u, t) < 4. \quad (13)$$

Solving the quadratic equation

$$(a(1, t) + a(1, t)^{-1})^2 \cos^2(t) - 2(a(1, t) + a(1, t)^{-1}) \sin(t) \cos(t)x + \sin^2(t)x^2 = 4 \quad (14)$$

we get

$$x = \frac{2(a(1, t) + a(1, t)^{-1}) \cos(t) \sin(t) \pm 4 \sin(t)}{2 \sin^2(t)} = \frac{(a(1, t) + a(1, t)^{-1}) \cos(t) \pm 2}{\sin(t)}.$$

Comparing (13) and (14) one obtains

$$\left( b(u, t) - \frac{(a(1, t) + a(1, t)^{-1}) \cos(t) - 2}{\sin(t)} \right) \left( b(u, t) - \frac{(a(1, t) + a(1, t)^{-1}) \cos(t) + 2}{\sin(t)} \right) < 0$$

which yields inequalities (10) and (11).  $\square$

**Proposition 7.** *Let  $Q^*$  be the multiplicative loop of a locally compact quasifield  $Q$  of dimension 2 containing a 1-dimensional compact normal subloop. The quasifield  $Q$  is the field  $\mathbb{C}$  of complex numbers if and only if  $\mathcal{T}$  is a normal subset in the set of all left translations of  $Q^*$ .*

*Proof.* If  $Q$  is the field of complex numbers, then  $Q^*$  is the group  $\text{SO}_2(\mathbb{R}) \times \mathbb{R}$  and the assertion is true. If the set  $\mathcal{T}$  is a normal subset in the set of the left translations of a proper loop  $Q^*$ , then it is normal in the connected component  $\text{GL}_2^+(\mathbb{R})$  of the group  $\text{GL}_2(\mathbb{R})$  because  $\text{GL}_2^+(\mathbb{R})$  is the group topologically generated by the left translations of  $Q^*$  (cf. [14], Section 29, p. 345). If  $\mathcal{T}$  is normal in  $\text{GL}_2^+(\mathbb{R})$ , then for  $D = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}$  one has that

$$D^{-1} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} D D^{-1} \begin{pmatrix} a(1, t) & b(1, t) \\ 0 & a^{-1}(1, t) \end{pmatrix} D =$$

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} D^{-1} \begin{pmatrix} a(1, t) & b(1, t) \\ 0 & a^{-1}(1, t) \end{pmatrix} D = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1, t) & b(1, t) \\ 0 & a^{-1}(1, t) \end{pmatrix}$$

is satisfied for all  $\varphi \in [0, 2\pi)$  if and only if  $a(1, t) = 1$  and  $b(1, t) = 0$  or equivalently  $\mathcal{T} = \text{SO}_2(\mathbb{R})$ . But the compact group  $\text{SO}_2(\mathbb{R})$  is not normal in  $\text{GL}_2^+(\mathbb{R})$ . Hence  $Q^*$  is not proper and the assertion follows.  $\square$

**Lemma 8.** *If the multiplicative loop  $Q^*$  of a 2-dimensional locally compact connected topological quasifield  $Q$  is not quasi-simple, then the set  $\mathcal{K} = \{M(r \cos(k\pi)a(r, k\pi), 0); r > 0, k \in \{0, 1\}\}$  of the left translations of  $Q^*$  belonging to the kernel  $K_r$  of  $Q$  has the form  $\mathcal{K} = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\}$ , which is a normal subgroup of the set  $\Lambda_{Q^*}$  of all left translations of  $Q^*$ .*

*Proof.* If  $Q$  is the field of complex numbers, then the assertion is true. If the loop  $Q^*$  is proper and not quasi-simple, then by Lemma 1.7, p. 19, in [14], the left translations of a normal subloop of  $Q^*$  generate a normal subgroup  $N$  of  $\text{GL}_2^+(\mathbb{R})$  which is the group topologically generated by all left translations

of  $Q^*$ . Hence the set  $\Lambda_{Q^*}$  of the left translations of  $Q^*$  must contain the group  $C = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, 0 < u \in \mathbb{R} \right\}$  as a normal subgroup. It follows that the set  $\mathcal{K}$  of the left translations corresponding to the elements of the kernel  $K_r$  of  $Q$  contained in  $\Lambda_{Q^*}$  has the form given in the assertion and  $\mathcal{K} \cap \Lambda_{Q^*}$  is a normal subgroup of  $\Lambda_{Q^*}$ .  $\square$

Assume that the set  $\mathcal{K}$  of the left translations of the loop  $Q^*$  having  $(1, 0)^t$  as identity corresponding to the elements of the kernel  $K_r$  of  $Q$  has the form given in Lemma 8. According to (3) the element

$$\begin{pmatrix} ra(r, \varphi) \cos \varphi & rb(r, \varphi) \cos \varphi + ra^{-1}(r, \varphi) \sin \varphi \\ -ra(r, \varphi) \sin \varphi & -rb(r, \varphi) \sin \varphi + ra^{-1}(r, \varphi) \cos \varphi \end{pmatrix}$$

corresponds to the left translation of  $(ra(r, \varphi) \cos \varphi, -ra(r, \varphi) \sin \varphi)^t$ ,  $r > 0$ ,  $\varphi \in [0, 2\pi)$ . Let  $N^*$  be the subgroup of  $Q^*$  corresponding to the normal subgroup  $\mathcal{K}$  of  $\Lambda_{Q^*}$ . We show that  $N^* := \{(\hat{s}, 0)^t, \hat{s} \in \mathbb{R} \setminus \{0\}\}$  is normal in  $Q^*$ . For all elements  $x := (\cos \varphi, -\sin \varphi)^t$ ,  $y := (u, v)^t$  of  $Q^*$  the condition  $(N^* * x) * y = N^* * (x * y)$  of (1) is satisfied if and only if we have

$$\left[ \begin{pmatrix} \hat{s} \\ 0 \end{pmatrix} * \begin{pmatrix} \cos \varphi \\ -\sin \varphi \end{pmatrix} \right] * \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \hat{s}' \\ 0 \end{pmatrix} * \left[ \begin{pmatrix} \cos \varphi \\ -\sin \varphi \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix} \right]$$

for all  $\varphi \in [0, 2\pi)$ ,  $(u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\}$  with suitable  $\hat{s}, \hat{s}' \in \mathbb{R} \setminus \{0\}$  such that  $\hat{s} = s \cos(k\pi)$ ,  $\hat{s}' = s' \cos(k\pi)$ , where  $s, s' > 0$ ,  $k \in \{0, 1\}$ . This is the case precisely if one has

$$\begin{pmatrix} usa(s, \varphi + k\pi) \cos(\varphi + k\pi) + vsb(s, \varphi + k\pi) \cos(\varphi + k\pi) + vsa^{-1}(s, \varphi + k\pi) \sin(\varphi + k\pi) \\ -usa(s, \varphi + k\pi) \sin(\varphi + k\pi) - vsb(s, \varphi + k\pi) \sin(\varphi + k\pi) + vsa^{-1}(s, \varphi + k\pi) \cos(\varphi + k\pi) \end{pmatrix} = \\ \begin{pmatrix} s' \cos(k\pi)(ua(1, \varphi) \cos \varphi + vb(1, \varphi) \cos \varphi + va^{-1}(1, \varphi) \sin \varphi) \\ s' \cos(k\pi)(-ua(1, \varphi) \sin \varphi - vb(1, \varphi) \sin \varphi + va^{-1}(1, \varphi) \cos \varphi) \end{pmatrix}$$

or equivalently for all  $\varphi \in [0, 2\pi)$ ,  $(u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\}$  we have

$$[(ua(s, \varphi + k\pi) + vb(s, \varphi + k\pi)) \cos(\varphi + k\pi) + va^{-1}(s, \varphi + k\pi) \sin(\varphi + k\pi)] \cdot [(-ua(1, \varphi) - vb(1, \varphi)) \sin \varphi + va^{-1}(1, \varphi) \cos \varphi] =$$

$$[(-ua(s, \varphi + k\pi) - vb(s, \varphi + k\pi)) \sin(\varphi + k\pi) + va^{-1}(s, \varphi + k\pi) \cos(\varphi + k\pi)] \cdot [(ua(1, \varphi) + vb(1, \varphi)) \cos \varphi + va^{-1}(1, \varphi) \sin \varphi].$$

The last equation holds if and only if one has

$$(a(s, \varphi + k\pi)a^{-1}(1, \varphi) - a^{-1}(s, \varphi + k\pi)a(1, \varphi))uv + (b(s, \varphi + k\pi)a^{-1}(1, \varphi) - a^{-1}(s, \varphi + k\pi)b(1, \varphi))v^2 = 0.$$

Therefore we obtain  $a(s, \varphi + k\pi)a^{-1}(1, \varphi) - a^{-1}(s, \varphi + k\pi)a(1, \varphi) = 0$  and  $b(s, \varphi + k\pi)a^{-1}(1, \varphi) - a^{-1}(s, \varphi + k\pi)b(1, \varphi) = 0$ . As  $a(s, \varphi)$  is positive we have  $a(s, \varphi + k\pi) = a(1, \varphi)$ ,  $b(s, \varphi + k\pi) = b(1, \varphi)$  for all  $s > 0$ ,  $\varphi \in [0, 2\pi)$ ,  $k \in \{0, 1\}$ . By (1) the group  $N^*$  is a normal subgroup of  $Q^*$  if and only if for all  $\varphi \in [0, 2\pi)$ ,  $(u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\}$  one has

$$\left[ \begin{pmatrix} \cos \varphi \\ -\sin \varphi \end{pmatrix} * \begin{pmatrix} \hat{s} \\ 0 \end{pmatrix} \right] * \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos \varphi \\ -\sin \varphi \end{pmatrix} * \left[ \begin{pmatrix} \hat{s}' \\ 0 \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix} \right] \quad \text{or}$$

$$\begin{pmatrix} sa(1, \varphi)a(1, \varphi) \cos(\varphi + k\pi) & sa(1, \varphi)b(1, \varphi) \cos(\varphi + k\pi) + s \sin(\varphi + k\pi) \\ -sa(1, \varphi)a(1, \varphi) \sin(\varphi + k\pi) & -sa(1, \varphi)b(1, \varphi) \sin(\varphi + k\pi) + s \cos(\varphi + k\pi) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \\ \begin{pmatrix} a(1, \varphi) \cos \varphi & b(1, \varphi) \cos \varphi + a^{-1}(1, \varphi) \sin \varphi \\ -a(1, \varphi) \sin \varphi & -b(1, \varphi) \sin \varphi + a^{-1}(1, \varphi) \cos \varphi \end{pmatrix} \begin{pmatrix} s' \cos(k\pi)u \\ s' \cos(k\pi)v \end{pmatrix}$$

for suitable  $\hat{s}, \hat{s}' \in \mathbb{R} \setminus \{0\}$  such that  $\hat{s} = s \cos(k\pi)$ ,  $\hat{s}' = s' \cos(k\pi)$ , where  $s, s' > 0$ ,  $k \in \{0, 1\}$ . This is equivalent to

$$\begin{pmatrix} sua(1, \varphi)^2 \cos(\varphi + k\pi) + sv[a(1, \varphi)b(1, \varphi) \cos(\varphi + k\pi) + \sin(\varphi + k\pi)] \\ -sua(1, \varphi)^2 \sin(\varphi + k\pi) + sv[-a(1, \varphi)b(1, \varphi) \sin(\varphi + k\pi) + \cos(\varphi + k\pi)] \end{pmatrix} = \\ \begin{pmatrix} us'a(1, \varphi) \cos(\varphi + k\pi) + s'v[b(1, \varphi) \cos(\varphi + k\pi) + a^{-1}(1, \varphi) \sin(\varphi + k\pi)] \\ -us'a(1, \varphi) \sin(\varphi + k\pi) + s'v[-b(1, \varphi) \sin(\varphi + k\pi) + a^{-1}(1, \varphi) \cos(\varphi + k\pi)] \end{pmatrix}.$$

A direct computation yields that the above equality is true for all  $(u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ ,  $\varphi \in [0, 2\pi)$ ,  $k \in \{0, 1\}$ .

Using Proposition 7, Lemma 8 and the discussion above we have the following

**Theorem 9.** *The multiplicative loop  $Q^*$  of a locally compact 2-dimensional quasifield  $Q$  with  $(1, 0)^t$  as identity of  $Q^*$  is not quasi-simple if and only if for all  $r > 0$ ,  $\varphi \in [0, 2\pi)$ ,  $k \in \{0, 1\}$  one has  $a(r, k\pi) = 1$ ,  $b(r, k\pi) = 0$ ,  $a(r, \varphi + k\pi) = a(1, \varphi)$  and  $b(r, \varphi + k\pi) = b(1, \varphi)$ . Then  $Q^*$  is a split extension of a 1-dimensional normal subgroup  $N^*$  by a subloop homeomorphic to the 1-sphere. Moreover, one has*

- a)  $N^*$  is isomorphic to  $\mathbb{R} \times Z_2$ , where  $Z_2$  is the group of order 2.
- b) This extension is the direct product precisely if  $Q$  is the field  $\mathbb{C}$ .

*Proof.* We have only to prove a) and b). According to Lemma 8 and the above discussion the only possibility for a normal subloop of positive dimension is the group  $N^*$ . The intersection of a compact subloop of  $Q^*$  with  $N^*$  has cardinality 2 (cf. Proposition 5 and Lemma 8). Hence the claim a) is proved. The claim of b) follows from Proposition 7.  $\square$

From Proposition 7 and Theorem 9 it follows:

**Corollary 10.** *The multiplicative loop  $Q^*$  of a locally compact 2-dimensional quasifield  $Q$  with  $(1, 0)^t$  as identity of  $Q^*$  is the direct product of the group  $\mathbb{R}$  and a subloop homeomorphic to the 1-sphere if and only if  $Q$  is the field of complex numbers.*

The set  $\Lambda_{Q^*}$  of the left translations of  $Q^*$  with a normal subloop of positive dimension and with  $(1, 0)^t$  as identity can be written into the form

$$\left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ua(1, t) & ub(1, t) \\ 0 & ua^{-1}(1, t) \end{pmatrix}, u > 0, t \in [0, 2\pi) \right\} \quad (15)$$

with  $a(1, k\pi) = 1$ ,  $b(1, k\pi) = 0$ ,  $k \in \{0, 1\}$ .

## 5. Decomposable multiplicative loops of 2-dimensional quasifields

**Definition 1.** We call the multiplicative loop  $Q^*$  of a locally compact connected topological 2-dimensional quasifield  $Q$  decomposable, if the set of all left translations of  $Q^*$  is a product  $\mathcal{TK}$  with  $|\mathcal{T} \cap \mathcal{K}| \leq 2$ , where  $\mathcal{T}$  is the set of all left translations of a 1-dimensional compact loop of the form (8) and  $\mathcal{K}$  is the set of all left translations of  $Q^*$  belonging to the kernel  $K_r$  of  $Q$ .

If the loop  $Q^*$  is decomposable, then it contains compact subloops for any  $u > 0$  corresponding to the section (7). From now on we choose  $u = 1$ . Then the compact loop  $F_u$  of  $Q^*$  has identity  $(1, 0)^t$  and one has

$$\begin{aligned} & \begin{pmatrix} \cos ta(1, t) & \cos tb(1, t) + \sin ta^{-1}(1, t) \\ -\sin ta(1, t) & -\sin tb(1, t) + \cos ta^{-1}(1, t) \end{pmatrix} \left[ \begin{pmatrix} r \cos(k\pi)a(r, k\pi) & r \cos(k\pi)b(r, k\pi) \\ 0 & r \cos(k\pi)a^{-1}(r, k\pi) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = \\ & = \begin{pmatrix} r \cos(t + k\pi)a(r, t + k\pi) & r \cos(t + k\pi)b(r, t + k\pi) + r \sin(t + k\pi)a^{-1}(r, t + k\pi) \\ -r \sin(t + k\pi)a(r, t + k\pi) & -r \sin(t + k\pi)b(r, t + k\pi) + r \cos(t + k\pi)a^{-1}(r, t + k\pi) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \quad (16)$$

Equation (16) yields that  $a(r, t + k\pi) = a(1, t)a(r, k\pi)$ .

Now we give sufficient and necessary conditions for the loop  $Q^*$  to be decomposable.

**Proposition 11.** The multiplicative loop  $Q^*$  of a locally compact connected topological 2-dimensional quasifield  $Q$  with  $(1, 0)^t$  as identity of  $Q^*$  is decomposable if and only if for all  $r > 0$ ,  $t \in [0, 2\pi)$ ,  $k \in \{0, 1\}$  one has

$$a(r, t + k\pi) = a(1, t)a(r, k\pi), \quad b(r, t + k\pi) = a(1, t)b(r, k\pi) + a^{-1}(r, k\pi)b(1, t).$$

*Proof.* The point  $(x, y)^t$  is the image of the point  $(1, 0)^t$  under the linear mapping  $M_{(x,y)}$  and the set  $\{M_{(x,y)}; (x, y)^t \in Q^*\}$  acts sharply transitively on  $Q^*$ . The matrix equation

$$\begin{aligned} & \begin{pmatrix} \cos ta(1, t) & \cos tb(1, t) + \sin ta^{-1}(1, t) \\ -\sin ta(1, t) & -\sin tb(1, t) + \cos ta^{-1}(1, t) \end{pmatrix} \left[ \begin{pmatrix} r \cos(k\pi)a(r, k\pi) & r \cos(k\pi)b(r, k\pi) \\ 0 & r \cos(k\pi)a^{-1}(r, k\pi) \end{pmatrix} \begin{pmatrix} u \cos \varphi a(u, \varphi) \\ -u \sin \varphi a(u, \varphi) \end{pmatrix} \right] = \\ & = \begin{pmatrix} r \cos(t + k\pi)a(r, t + k\pi) & r \cos(t + k\pi)b(r, t + k\pi) + r \sin(t + k\pi)a^{-1}(r, t + k\pi) \\ -r \sin(t + k\pi)a(r, t + k\pi) & -r \sin(t + k\pi)b(r, t + k\pi) + r \cos(t + k\pi)a^{-1}(r, t + k\pi) \end{pmatrix} \begin{pmatrix} u \cos \varphi a(u, \varphi) \\ -u \sin \varphi a(u, \varphi) \end{pmatrix} \end{aligned} \quad (17)$$

holds precisely if the identities of the assertion are satisfied.  $\square$

If  $Q^*$  is decomposable such that its compact subloop has identity  $(1, 0)^t$ , then  $|\mathcal{T} \cap \mathcal{K}| = 2$  because one has  $\mathcal{T} \cap \mathcal{K} = \left\{ I, \begin{pmatrix} -1 & -b(1, \pi) \\ 0 & -1 \end{pmatrix} \right\}$ . In this case the set of all left translations of  $Q^*$  is a product  $\mathcal{TW}$  with  $\mathcal{T} \cap \mathcal{W} = I$ , where  $\mathcal{W}$  is the set of all left translations corresponding to the connected component of the kernel  $K_r$  of  $Q$ .

**Theorem 12.** If the multiplicative loop  $Q^*$  of a locally compact connected topological 2-dimensional quasifield  $Q$  with  $(1, 0)^t$  as identity of  $Q^*$  is not quasi-simple, then  $Q^*$  is decomposable.

*Proof.* By Theorem 9 the loop  $Q^*$  is not quasi-simple if and only if for all  $r > 0$ ,  $t \in [0, 2\pi)$ ,  $k \in \{0, 1\}$  one has  $a(r, k\pi) = 1$ ,  $b(r, k\pi) = 0$ ,  $a(r, t + k\pi) = a(1, t)$  and  $b(r, t + k\pi) = b(1, t)$ . Therefore the identities given in the assertion of Proposition 11 are satisfied.  $\square$

**Proposition 13.** *The  $\mathcal{C}^1$ -differentiable multiplicative loop  $Q^*$  of a locally compact connected topological 2-dimensional quasifield  $Q$  with  $(1, 0)^t$  as identity of  $Q^*$  is decomposable precisely if for the inverse function  $\bar{a}(1, t) = a^{-1}(1, t)$  and for  $\bar{b}(1, t) = -b(1, t)$  the differential inequalities*

$$\bar{a}'^2(1, t) + \bar{b}(1, t)\bar{a}'(1, t) + \bar{b}'(1, t)\bar{a}(1, t) - \bar{a}^2(1, t) < 0, \quad \bar{b}'(1, 0) < 1 - \bar{a}'^2(1, 0) \quad (18)$$

are satisfied.

*Proof.* If  $Q^*$  is a  $\mathcal{C}^1$ -differentiable multiplicative loop of a quasifield  $Q$  with  $(1, 0)^t$  as identity, then for  $u = 1$  the continuously differentiable functions  $\bar{a}_u(t) := a(u, 0)a^{-1}(u, t)$ ,  $\bar{b}_u(t) := -b(u, t)$  satisfy the conditions in (6). The set of all left translations of  $Q^*$  is a product  $\mathcal{TK}$  if and only if one has  $a(u, t + k\pi) = a(u, k\pi)a(1, t)$  and  $b(u, t + k\pi) = a(1, t)b(u, k\pi) + a^{-1}(u, k\pi)b(1, t)$  for all  $u > 0$ ,  $t \in [0, 2\pi)$ ,  $k \in \{0, 1\}$  (cf. Proposition 11). Putting this into (6) we get for  $u = 1$

$$a'^2(1, t) + b(1, t)a'(1, t)a^2(1, t) - b'(1, t)a^3(1, t) - a^2(1, t) < 0 \text{ and} \\ b'(1, 0) > a'^2(1, 0) - 1 \quad (19)$$

since  $a(1, k\pi) = 1$ ,  $k \in \{0, 1\}$ . Inequalities (19) are equivalent to the inequalities (18) with  $\bar{a}(1, t) = a^{-1}(1, t)$  and  $\bar{b}(1, t) = -b(1, t)$ .  $\square$

**Corollary 14.** *Let  $T$  be any 1-dimensional  $\mathcal{C}^1$ -differentiable connected compact loop such that the set  $\mathcal{T}$  of its left translations has the form (8) and let  $\mathcal{K}$  be any set of matrices of the form*

$$\mathcal{K} = \left\{ \begin{pmatrix} u \cos(k\pi)a(u, k\pi) & u \cos(k\pi)b(u, k\pi) \\ 0 & u \cos(k\pi)a^{-1}(u, k\pi) \end{pmatrix}, u > 0, k \in \{0, 1\} \right\},$$

where  $a(u, k\pi) > 0$  and  $b(u, k\pi)$  are continuously differentiable functions such that  $ua(u, 0)$ ,  $-ua(u, \pi)$  are strictly monotone. Then the product  $\mathcal{TK}$  is the set of all left translations of a  $\mathcal{C}^1$ -differentiable decomposable multiplicative loop  $Q^*$  of a 2-dimensional locally compact connected quasifield  $Q$  with  $(1, 0)^t$  as identity of  $Q^*$ .

*Proof.* Since one has

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1, t) & b(1, t) \\ 0 & a(1, t)^{-1} \end{pmatrix} \begin{pmatrix} u \cos(k\pi)a(u, k\pi) & u \cos(k\pi)b(u, k\pi) \\ 0 & u \cos(k\pi)a^{-1}(u, k\pi) \end{pmatrix} =$$

$$\begin{pmatrix} \cos(t+k\pi) & \sin(t+k\pi) \\ -\sin(t+k\pi) & \cos(t+k\pi) \end{pmatrix} \begin{pmatrix} ua(u, k\pi)a(1, t) & ub(u, k\pi)a(1, t) + ub(1, t)a^{-1}(u, k\pi) \\ 0 & ua^{-1}(u, k\pi)a(1, t)^{-1} \end{pmatrix} = \\ \begin{pmatrix} \cos(t+k\pi) & \sin(t+k\pi) \\ -\sin(t+k\pi) & \cos(t+k\pi) \end{pmatrix} \begin{pmatrix} ua(u, t+k\pi) & ub(u, t+k\pi) \\ 0 & ua^{-1}(u, t+k\pi) \end{pmatrix},$$

for the continuously differentiable functions  $a(1, t)$ ,  $b(1, t)$  the inequalities (19) hold and  $a(1, k\pi) = 1$ ,  $k \in \{0, 1\}$ , for  $u = 1$  the continuously differentiable functions  $\bar{a}_u(t+k\pi) = a(u, 0)a^{-1}(u, t+k\pi) = a(u, 0)a^{-1}(u, k\pi)a^{-1}(1, t)$ ,  $\bar{b}_u(t+k\pi) = -b(u, t+k\pi) = -b(u, k\pi)a(1, t) - b(1, t)a^{-1}(u, k\pi)$  satisfy inequalities (6). Hence the product  $\mathcal{TK}$  given in the assertion is the image of a  $\mathcal{C}^1$ -differentiable section of a multiplicative loop  $Q^*$  of a quasifield  $Q$ .  $\square$

**Proposition 15.** *The set  $\Lambda_{Q^*}$  of all left translations of the multiplicative loop  $Q^*$  for a locally compact connected topological 2-dimensional quasifield  $Q$  with  $(1, 0)^t$  as identity of  $Q^*$  contains the group  $\text{SO}_2(\mathbb{R})$  if and only if  $\Lambda_{Q^*}$  has the form*

$$\Lambda_{Q^*} = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ua(u, 0) & ub(u, 0) \\ 0 & ua^{-1}(u, 0) \end{pmatrix}, u > 0, t \in [0, 2\pi) \right\} \quad (20)$$

where  $a(u, 0)$ ,  $b(u, 0)$  are arbitrary continuous functions with  $a(u, 0) > 0$  such that  $ua(u, 0)$  is strictly monotone. In this case  $Q^*$  is decomposable.

*Proof.* If the set  $\Lambda_{Q^*}$  contains the group  $\text{SO}_2(\mathbb{R})$ , then for each fixed  $u > 0$  the function  $a(u, t)$  is constant with value 1 and the function  $b(u, t)$  is constant with value 0. So the functions  $a(u, t) = a(u, 0)$ ,  $b(u, t) = b(u, 0)$  do not depend on the variable  $t$ . Hence the identities in Proposition 11 are satisfied and the set  $\Lambda_{Q^*}$  has the form  $\mathcal{TW}$  as in the assertion.

Conversely, if  $ua(u, 0)$  is a strictly monotone continuous function, then for arbitrary continuous functions  $a(u, 0)$ ,  $b(u, 0)$  with  $a(u, 0) > 0$  the set given by (20) is the set  $\Lambda_{Q^*}$  of all left translations of the multiplicative loop  $Q^*$  of a locally compact quasifield such that  $\Lambda_{Q^*}$  contains the group  $\text{SO}_2(\mathbb{R})$  and  $Q^*$  has identity  $(1, 0)^t$ . Hence  $Q^*$  is decomposable and the assertion is proved.  $\square$

## 6. Betten's classification of 4-dimensional translation planes

Using 2-dimensional spreads, Betten in [3], [4], [5], [6], [7], [8], see also [13] and [15], has classified all locally compact 4-dimensional translation planes which admit an at least 7-dimensional collineation group. His normalized 2-dimensional spreads are images of sharply transitive sections  $\sigma' : G/H' \rightarrow G$ , where  $G$  is the connected component of the group  $\text{GL}_2(\mathbb{R})$ ,  $H'$  is the subgroup  $\left\{ \begin{pmatrix} 1 & c \\ 0 & d \end{pmatrix}, d > 0, c \in \mathbb{R} \right\}$  (cf. [2], [3]) and  $\sigma'(G/H')$  consists of the elements

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ra(r, t) & 0 \\ 0 & r^{-1}a^{-1}(r, t) \end{pmatrix} \begin{pmatrix} 1 & b(r, t)a^{-1}(r, t) \\ 0 & r^2 \end{pmatrix}.$$



With respect to the stabilizer  $H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}$  the sharply transitive section  $\sigma'$  transforms into a sharply transitive section  $\sigma : G/H \rightarrow G$  given by (2), because the elements of  $\sigma'(G/H')$  coincide with

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} a(r, t) & b(r, t) \\ 0 & a^{-1}(r, t) \end{pmatrix}.$$

**Proposition 16.** *Let  $\mathcal{A}$  be a 4-dimensional non-desarguesian translation plane admitting an 8-dimensional collineation group such that  $\mathcal{A}$  is coordinatized by the locally compact topological quasifield  $Q$ . Then the multiplicative loop  $Q^*$  can be given by one of the following sets  $\Lambda_{Q^*}$  of the left translations of  $Q^*$ :*

a)  $\Lambda_{Q^*}$  has the form (15) with  $a(1, t) = 1$  and  $b(1, t) = 0$  for  $0 \leq t \leq \pi$ ,  $a(1, t) = 1/\sqrt{\cos^2 t + \frac{\sin^2 t}{w}}$  and  $b(1, t) = a(1, t) \frac{1-w}{w} \sin t \cos t$  for  $\pi < t < 2\pi$ . The quasifields  $Q_w$ ,  $w > 1$ , correspond to a one-parameter family of planes  $\mathcal{A}_w$ .

b)  $\Lambda_{Q^*}$  is the range of the section given by (2) such that for  $\alpha \geq \frac{-3\beta^2}{4}$  one has  $a(r, t) = \frac{\sqrt{\alpha^2 + \beta^2}}{\alpha + \beta^2}$  and  $b(r, t) = \frac{\beta(-\alpha+1)}{\sqrt{\alpha^2 + \beta^2}}$  with  $r \cos(t) = \alpha \frac{\alpha + \beta^2}{\sqrt{\alpha^2 + \beta^2}}$ ,  $r \sin(t) = -\beta \frac{\alpha + \beta^2}{\sqrt{\alpha^2 + \beta^2}}$ .

For  $\alpha < \frac{-3\beta^2}{4}$  we have  $a(r, t) = 3\sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2}}$  and  $b(r, t) = \beta\sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2}} + \frac{\beta\alpha}{3\sqrt{\alpha^2(\alpha^2 + \beta^2)}}$  with  $r \cos(t) = \frac{\alpha}{3}\sqrt{\frac{\alpha^2}{\alpha^2 + \beta^2}}$ ,  $r \sin(t) = -\frac{\beta}{3}\sqrt{\frac{\alpha^2}{\alpha^2 + \beta^2}}$ . The quasifield  $Q$  coordinatizes a single plane.

c)  $\Lambda_{Q^*}$  is the range of the section given by (2) such that  $a(r, t) = \sqrt{\frac{v^2 + s^2}{\frac{s^4}{3} + s^2 v + v^2}}$ ,

$$b(r, t) = \frac{-\frac{s^3 v}{3} + s^3 + sv}{\sqrt{(\frac{s^4}{3} + s^2 v + v^2)(s^2 + v^2)}} \text{ with}$$

$$r \cos(t) = v \sqrt{\frac{\frac{s^4}{3} + s^2 v + v^2}{s^2 + v^2}}, \quad r \sin(t) = -s \sqrt{\frac{\frac{s^4}{3} + s^2 v + v^2}{s^2 + v^2}}.$$

The quasifield  $Q$  coordinatizes a single plane.

In case a) the multiplicative loop  $Q_w^*$  is decomposable and a split extension of the normal subgroup  $\widetilde{N}^* \cong \mathbb{R}$  corresponding to the connected component of

$$\widetilde{\mathcal{K}} = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\} \text{ with a subloop homeomorphic to the 1-sphere.}$$

In case c) the set  $\mathcal{K}$  of the left translations of  $Q^*$  corresponding to the kernel  $K_r$  of the quasifield  $Q$  has the form  $\widetilde{\mathcal{K}}$ . In case b) the set  $\mathcal{K}$  has the form

$$\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \alpha > 0 \right\} \cup \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \frac{\alpha}{9} \end{pmatrix}, \alpha < 0 \right\}.$$

In cases b) and c) the multiplicative loops  $Q^*$  are not decomposable and quasi-simple.

*Proof.* If the translation complement of  $\mathcal{A}$  is the group  $\mathrm{GL}_2(\mathbb{R})$  and acts reducibly on  $\mathbb{R}^4$ , then one obtains the one-parameter family  $\mathcal{A}_w$ ,  $w > 1$ , of the non-desarguesian translation planes corresponding to the following spreads:

$$\{S\} \cup \left\{ \begin{pmatrix} s & -v \\ v & s \end{pmatrix}, s, v \in \mathbb{R}, v \geq 0 \right\} \cup \left\{ \begin{pmatrix} s & \frac{-v}{w} \\ v & s \end{pmatrix}, s, v \in \mathbb{R}, v < 0 \right\},$$

$w > 1$  (cf. [3], Satz 5, p. 144). Any such spread coincides with the set  $\Lambda_{Q^*}$  in (15) with  $a(1, t)$  and  $b(1, t)$  as in assertion a). By Theorem 9 the multiplicative loop  $Q_w^*$  is a split extension of a normal subgroup  $N^*$  with a 1-dimensional compact loop. By Theorem 12 the loop  $Q_w^*$  is decomposable. Hence  $\widetilde{N}^*$  has the form as in the assertion.

If the translation complement  $\mathrm{GL}_2(\mathbb{R})$  acts irreducibly on  $\mathbb{R}^4$ , then one obtains a single plane  $\mathcal{A}$  generated by the spread

$$\{S\} \cup \left\{ \begin{pmatrix} \alpha & -\alpha\beta - \beta^3 \\ \beta & \alpha + \beta^2 \end{pmatrix}, \alpha, \beta \in \mathbb{R}, \alpha \geq \frac{-3\beta^2}{4} \right\} \cup \left\{ \begin{pmatrix} \alpha & \frac{1}{3}\alpha\beta \\ \beta & \frac{\alpha}{9} + \frac{\beta^2}{3} \end{pmatrix}, \alpha, \beta \in \mathbb{R}, \alpha < \frac{-3\beta^2}{4} \right\}, \quad (21)$$

(cf. [5], Satz, p. 553).

If the translation complement is solvable, then one gets a single plane  $\mathcal{A}$  generated by the spread

$$\{S\} \cup \left\{ \begin{pmatrix} v & -\frac{s^3}{3} \\ s & s^2 + v \end{pmatrix}, s, v \in \mathbb{R} \right\}, \quad (22)$$

(cf. [4], Satz 2 (b), p. 331).

The spread (21), respectively (22) coincides with the image of the section  $\sigma$  in (2) with the well defined functions  $a(r, t)$  and  $b(r, t)$  given in assertion b), respectively c). In case c) one has  $a(r, k\pi) = 1$ ,  $b(r, k\pi) = 0$ , for all  $r > 0$ ,  $k \in \{0, 1\}$  hence Remark 1 gives the form  $\mathcal{K}$  of the assertion. In case b) we have  $a(r, 0) = 1$ ,  $a(r, \pi) = 3$ ,  $b(r, l\pi) = 0$ , for all  $r > 0$ ,  $l \in \{0, 1\}$ . These give the form  $\mathcal{K}$  of the assertion.

For decomposable  $Q^*$ , the identity  $a(r, t + k\pi) = a(1, t)a(r, k\pi)$  holds for all  $r > 0$ ,  $t \in [0, 2\pi)$ ,  $k \in \{0, 1\}$  (cf. Proposition 11). In case b) for  $-3 \leq \alpha \leq 1$  one has  $a(1, t) = \sqrt{\alpha^2 - \alpha + 1}$  which yields a contradiction. In case c) we have  $a(r, \frac{\pi}{4} + k\pi) = \sqrt{\frac{2}{1-s+\frac{s^2}{3}}}$ ,  $s \in \mathbb{R} \setminus \{0\}$  and the condition  $a(r, \frac{\pi}{4} + k\pi) = a(1, \frac{\pi}{4})$  gives a contradiction. Hence in both cases  $Q^*$  is not decomposable and therefore quasi-simple (cf. Theorem 12).  $\square$

If the translation complement of a 4-dimensional topological plane  $\mathcal{A}$  has dimension 3, then the point  $\infty$  of the line  $S = \{(0, 0, u, v), u, v \in \mathbb{R}\}$  is fixed under the seven-dimensional collineation group  $\Gamma$  of  $\mathcal{A}$ .

**Proposition 17.** *Let  $Q$  be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane  $\mathcal{A}$  such that the 7-dimensional collineation group  $\Gamma$  of  $\mathcal{A}$  acts transitively on the points of  $W \setminus \{\infty\}$ , where  $W$  is the translation axis of  $\mathcal{A}$  and the kernel of the action of the translation complement on the line  $S$  has dimension 1. Then the multiplicative loop  $Q^*$  can be given by one of the following sets  $\Lambda_{Q^*}$  of the left translations of  $Q^*$ :*

a)  $\Lambda_{Q^*}$  is the range of the section (2) such that

$$a(r, t) = \sqrt{\frac{s^2 + v^2}{s^2v + v^2 + \frac{s^4}{3} + s^2}} \quad \text{and} \quad b(r, t) = \frac{s^3 - \frac{s^3v}{3}}{\sqrt{(s^2v + v^2 + \frac{s^4}{3} + s^2)(s^2 + v^2)}}$$

with  $r \cos(t) = v \sqrt{\frac{s^2v + v^2 + \frac{s^4}{3} + s^2}{s^2 + v^2}}$ ,  $r \sin(t) = -s \sqrt{\frac{s^2v + v^2 + \frac{s^4}{3} + s^2}{s^2 + v^2}}$ . The quasifield  $Q$  corresponds to a single plane.

b)  $\Lambda_{Q^*}$  is the range of the section given by (2) such that

$$a(r, t) = \sqrt{\frac{v^2 + u^2 + 2\gamma^2(1 - \cos(u)) - 2v\gamma \sin(u) - 2\gamma u \cos(u) + 2\gamma u}{v^2 + u^2 - 2\gamma^2 + 2\gamma^2 \cos(u)}} \quad \text{and}$$

$$b(r, t) = \frac{-2u\gamma \sin u + 2v\gamma \cos u - 2v\gamma}{\sqrt{v^2 + u^2 + 2\gamma^2(1 - \cos u) - 2v\gamma \sin u - 2\gamma u \cos u + 2\gamma u \sqrt{v^2 + u^2 - 2\gamma^2(1 - \cos u)}}}$$

with

$$r \cos(t) = (v - \gamma \sin(u)) \sqrt{\frac{v^2 + u^2 - 2\gamma^2 + 2\gamma^2 \cos u}{v^2 + u^2 + 2\gamma^2(1 - \cos(u)) - 2v\gamma \sin(u) - 2\gamma u \cos(u) + 2\gamma u}}$$

$$r \sin(t) = (u - \gamma(\cos(u) - 1)) \sqrt{\frac{v^2 + u^2 - 2\gamma^2 + 2\gamma^2 \cos u}{v^2 + u^2 + 2\gamma^2(1 - \cos(u)) - 2v\gamma \sin(u) - 2\gamma u \cos(u) + 2\gamma u}}$$

The quasifields  $Q_\gamma$  coordinatize a one-parameter family of planes  $\mathcal{A}_\gamma$ ,  $0 < |\gamma| \leq 1$ .

In all cases the multiplicative loop  $Q^*$  is not decomposable and quasi-simple. The set  $\mathcal{K}$  of the left translations of  $Q^*$  corresponding to the kernel of the quasifield  $Q$  has the form  $\left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\}$ .

*Proof.* If the translation complement  $C$  leaves a 1-dimensional subspace of  $S$  invariant, then one obtains a single plane  $\mathcal{A}$  which corresponds to the following spread:

$$\{S\} \cup \left\{ \begin{pmatrix} v & -\frac{s^3}{3} - s \\ s & s^2 + v \end{pmatrix}, s, v \in \mathbb{R} \right\} \quad (23)$$

(cf. [18], 73.10., [4], pp. 330-331).

If the translation complement acts transitively on the 1-dimensional subspaces of  $S$ , then one gets a one-parameter family  $E_\gamma, 0 < |\gamma| \leq 1$ , of planes which are generated by the normalized spread

$$\{S\} \cup \left\{ \left( \begin{array}{cc} v - \gamma \sin u & u + \gamma(\cos u - 1) \\ \gamma(\cos u - 1) - u & v + \gamma \sin u \end{array} \right), u, v \in \mathbb{R} \right\}, \quad (24)$$

([8], Satz, p. 128, [13], Proposition 5.8). The spread (23), respectively (24) coincides with the image of the section  $\sigma$  in (2) such that the well defined functions  $a(r, t)$  and  $b(r, t)$  are given in assertion a), respectively b). Since in both cases one has  $a(r, k\pi) = 1, b(r, k\pi) = 0$ , for all  $r > 0, k \in \{0, 1\}$ , Remark 1 gives the form of  $\mathcal{K}$ . Moreover, in case a) one has  $a(r, \frac{\pi}{4} + k\pi) = \sqrt{\frac{2}{2+v+\frac{v^2}{3}}}, v \in \mathbb{R} \setminus \{0\}$ . In case b) for  $v = 1$  we get

$$a(r_j, t_j) = \sqrt{\frac{1+u^2+2\gamma^2(1-\cos u)-2\gamma \sin u-2\gamma u \cos u+2\gamma u}{1+u^2-2\gamma^2+2\gamma^2 \cos u}}, \quad a(1, t_j) = 1.$$

For decomposable  $Q^*$  one has  $a(r, t+k\pi) = a(1, t)a(r, k\pi)$  for all  $r \in \mathbb{R} \setminus \{0\}, t \in [0, 2\pi), k \in \{0, 1\}$  (cf. Proposition 11) which yields a contradiction. Thus in both cases  $Q^*$  is not decomposable and hence quasi-simple (cf. Theorem 12).  $\square$

**Proposition 18.** *Let  $Q$  be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane  $\mathcal{A}$  such that the translation complement  $C$  of the 7-dimensional collineation group  $\Gamma$  of  $\mathcal{A}$  has an orbit of dimension 1 on  $W \setminus \{0\}$ ,  $C$  leaves in the set of lines through the origin only  $S$  fixed and the kernel of its action on  $S$  has positive dimension. Then the multiplicative loop  $Q^*$  can be given by one of the following sets  $\Lambda_{Q^*}$  of the left translations of  $Q^*$ :*

a)  $\Lambda_{Q^*}$  is the range of the section (2) such that for  $\beta \geq 0$  one has

$$a(r, t) = \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 + z\alpha\beta \frac{1}{1+s} - w\beta \frac{2}{1+s}}} \text{ and } b(r, t) = \frac{w\alpha\beta \frac{1-s}{1+s} + \alpha\beta + z\beta \frac{2+s}{1+s}}{\sqrt{\alpha^2 + \beta^2} \sqrt{\alpha^2 + z\alpha\beta \frac{1}{1+s} - w\beta \frac{2}{1+s}}}$$

$$\text{with } r \cos(t) = \alpha \sqrt{\frac{\alpha^2 + z\alpha\beta \frac{1}{1+s} - w\beta \frac{2}{1+s}}{\alpha^2 + \beta^2}}, \quad r \sin(t) = -\beta \sqrt{\frac{\alpha^2 + z\alpha\beta \frac{1}{1+s} - w\beta \frac{2}{1+s}}{\alpha^2 + \beta^2}}.$$

For  $\beta < 0$  one gets

$$a(r', t) = \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 + q\alpha(-\beta) \frac{1}{1+s} + p(-\beta) \frac{2}{1+s}}} \text{ and } b(r', t) = \frac{p\alpha(-\beta) \frac{1-s}{1+s} + \alpha\beta - q(-\beta) \frac{2+s}{1+s}}{\sqrt{\alpha^2 + \beta^2} \sqrt{\alpha^2 + q\alpha(-\beta) \frac{1}{1+s} + p(-\beta) \frac{2}{1+s}}}$$

with

$$r' \cos(t) = \alpha \sqrt{\frac{\alpha^2 + q\alpha(-\beta) \frac{1}{1+s} + p(-\beta) \frac{2}{1+s}}{\alpha^2 + \beta^2}} \text{ and } r' \sin(t) = -\beta \sqrt{\frac{\alpha^2 + q\alpha(-\beta) \frac{1}{1+s} + p(-\beta) \frac{2}{1+s}}{\alpha^2 + \beta^2}}.$$

The quasifields  $Q_{s,w,z,p,q}$  coordinatize a family of planes  $\mathcal{A}_{s,w,z,p,q}$  such that the parameters  $s, w, z, p, q$  satisfy the conditions  $0 < s < 1$ ,  $z^2 + 4w(1 - s^2) \leq 0$ ,  $q^2 - 4p(1 - s^2) \leq 0$ .

b)  $\Lambda_{Q^*}$  is the range of the section (2) such that for  $\beta \geq 0$  we have

$$a(r, t) = \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 + z\alpha\beta - w\beta^2 + 2\alpha\beta \ln \beta + z\beta^2 \ln \beta + \beta^2 (\ln \beta)^2}} \text{ and}$$

$$b(r, t) = \frac{(w+1)\alpha\beta + z\beta^2 - z\alpha\beta \ln \beta - \alpha\beta (\ln \beta)^2 + 2\beta^2 \ln \beta}{\sqrt{\alpha^2 + \beta^2} \sqrt{\alpha^2 + z\alpha\beta + 2\alpha\beta \ln \beta - w\beta^2 + z\beta^2 \ln \beta + \beta^2 (\ln \beta)^2}}$$

with

$$r \cos(t) = \alpha \sqrt{\frac{\alpha^2 + z\alpha\beta - w\beta^2 + 2\alpha\beta \ln \beta + z\beta^2 \ln \beta + \beta^2 (\ln \beta)^2}{\alpha^2 + \beta^2}},$$

$$r \sin(t) = -\beta \sqrt{\frac{\alpha^2 + z\alpha\beta - w\beta^2 + 2\alpha\beta \ln \beta + z\beta^2 \ln \beta + \beta^2 (\ln \beta)^2}{\alpha^2 + \beta^2}}.$$

For  $\beta < 0$  we obtain

$$a(r', t) = \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 - q\alpha\beta + p\beta^2 + (2\alpha\beta - q\beta^2) \ln(-\beta) + \beta^2 (\ln(-\beta))^2}} \text{ and}$$

$$b(r', t) = \frac{(1-p)\alpha\beta - q\beta^2 + (2\beta^2 + q\alpha\beta) \ln(-\beta) - \alpha\beta (\ln(-\beta))^2}{\sqrt{\alpha^2 + \beta^2} \sqrt{\alpha^2 - q\alpha\beta + p\beta^2 + (2\alpha\beta - q\beta^2) \ln(-\beta) + \beta^2 (\ln(-\beta))^2}}$$

with

$$r' \cos(t) = \alpha \sqrt{\frac{\alpha^2 - q\alpha\beta + p\beta^2 + (2\alpha\beta - q\beta^2) \ln(-\beta) + \beta^2 (\ln(-\beta))^2}{\alpha^2 + \beta^2}},$$

$$r' \sin(t) = -\beta \sqrt{\frac{\alpha^2 - q\alpha\beta + p\beta^2 + (2\alpha\beta - q\beta^2) \ln(-\beta) + \beta^2 (\ln(-\beta))^2}{\alpha^2 + \beta^2}}.$$

The quasifields  $Q_{w,z,p,q}$  coordinatize a family of planes  $\mathcal{A}_{w,z,p,q}$  such that for the parameters  $w, z, p, q$  the relations  $(\frac{z}{2})^2 \leq -w - 1$ ,  $(\frac{q}{2})^2 \leq p - 1$  hold.

c)  $\Lambda_{Q^*}$  is the range of the section given by (2) such that  $a(r, k\pi) = 1$  and  $b(r, k\pi) = 0$ ,  $k \in \{0, 1\}$  with  $r = |\beta|$ .

For  $u \in \mathbb{R}$ ,  $\beta > 0$ , we get

$$a(r, t) = \sqrt{\frac{u^2 + \sin^2(l)(w^2 + 2zu + z^2) + \cos^2(l) - (2uw + 2u + 2z) \sin(l) \cos(l)}{u^2 + uz - w}},$$

$$b(r, t) = \frac{\cos^2(l)(2uw + 2u + 2z) + \sin(l) \cos(l)(1 - w^2 - z^2 - 2uz) - (u + z + uw)}{\sqrt{(u^2 + \sin^2(l)(w^2 + 2zu + z^2) + \cos^2(l) - (2uw + 2u + 2z) \sin(l) \cos(l))(u^2 + uz - w)}}$$

with

$$r \cos(t) = \beta (u - (w+1) \sin(l) \cos(l) + z \sin^2(l)) \sqrt{\frac{u^2 + uz - w}{u^2 + \sin^2(l)(w^2 + 2zu + z^2) + \cos^2(l) - (2uw + 2u + 2z) \sin(l) \cos(l)}},$$

$$r \sin(t) = \beta (w \sin^2(l) + z \sin(l) \cos(l) - \cos^2(l)) \sqrt{\frac{u^2 + uz - w}{u^2 + \sin^2(l)(w^2 + 2zu + z^2) + \cos^2(l) - (2uw + 2u + 2z) \sin(l) \cos(l)}},$$

where  $l = \frac{1}{k} \ln \beta$ . For  $u \in \mathbb{R}$ ,  $\beta < 0$  one obtains

$$a(r', t') = \sqrt{\frac{u^2 + \sin^2(t_1)(q^2 + 2qu + p^2) + \cos^2(t_1) + (2u + 2q - 2up) \sin(t_1) \cos(t_1)}{u^2 + uq + p}}$$

$$b(r', t') = \frac{\sin(l_1) \cos(l_1)(1 - 2uq - p^2 - q^2) + \sin^2(l_1)(2q + 2u - 2up) + (up - q - u)}{\sqrt{(u^2 + \sin^2(l_1)(q^2 + 2qu + p^2) + \cos^2(l_1) + (2q + 2u - 2up) \sin(l_1) \cos(l_1))(u^2 + uq + p)}}$$

with

$$\begin{aligned} r' \cos(t') &= \beta \left( (p-1) \sin(l_1) \cos(l_1) - q \sin^2(l_1) - u \right) \cdot \\ &\sqrt{\frac{u^2 + uq + p}{u^2 + \sin^2(l_1)(q^2 + 2qu + p^2) + \cos^2(l_1) + (2u + 2q - 2up) \sin(l_1) \cos(l_1)}}, \\ r' \sin(t') &= -\beta \left( \cos^2(l_1) + q \sin(l_1) \cos(l_1) + p \sin^2(l_1) \right) \cdot \\ &\sqrt{\frac{u^2 + uq + p}{u^2 + \sin^2(l_1)(q^2 + 2qu + p^2) + \cos^2(l_1) + (2u + 2q - 2up) \sin(l_1) \cos(l_1)}}, \end{aligned}$$

where  $l_1 = \frac{1}{k} \ln(-\beta)$ .

The quasifields  $Q_{k,w,z,p,q}$  coordinatize a family of planes  $\mathcal{A}_{k,w,z,p,q}$  such that for the parameters  $k, w, z, p, q$  one has  $k \neq 0$ ,  $(4+k^2)(z^2 + (w+1)^2) \leq k^2(1-w)^2$ ,  $(4+k^2)(q^2 + (p-1)^2) \leq k^2(p+1)^2$ ,  $(w, z, p, q) \neq (-1, 0, 1, 0)$ .

In all cases  $Q^*$  is not decomposable and quasi-simple. The set of the left translations of  $Q^*$  belonging to the kernel of  $Q$  is  $\mathcal{K} = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\}$ .

*Proof.* If the translation complement  $C$  fixes two 1-dimensional subspaces of  $S$ , then we have a family of translation planes  $\mathcal{A}_{s,w,z,p,q}$  such that the normalized spreads belonging to these planes are given as follows:

$$\{S\} \cup \left\{ \begin{pmatrix} \alpha & w\beta \frac{1-s}{1+s} \\ \beta & z\beta \frac{1}{1+s} + \alpha \end{pmatrix}, \alpha \in \mathbb{R}, \beta \geq 0 \right\} \cup \left\{ \begin{pmatrix} \alpha & p(-\beta) \frac{1-s}{1+s} \\ \beta & q(-\beta) \frac{1}{1+s} + \alpha \end{pmatrix}, \alpha \in \mathbb{R}, \beta < 0 \right\}, \quad (25)$$

(cf. [6], Satz 1, pp. 411-412).

If the translation complement  $C$  fixes only one 1-dimensional subspace of  $S$ , then there is a family of translation planes  $\mathcal{A}_{w,z,p,q}$  such that the corresponding normalized spreads have the form:

$$\{S\} \cup \left\{ \begin{pmatrix} \alpha & w\beta - z\beta \ln \beta - \beta (\ln \beta)^2 \\ \beta & \alpha + z\beta + 2\beta \ln \beta \end{pmatrix}, \alpha \in \mathbb{R}, \beta \geq 0 \right\} \cup \left\{ \begin{pmatrix} \alpha & -p\beta - \beta (\ln(-\beta))^2 + q\beta \ln(-\beta) \\ \beta & q(-\beta) + \alpha + 2\beta \ln(-\beta) \end{pmatrix}, \alpha \in \mathbb{R}, \beta < 0 \right\} \quad (26)$$

(cf. Satz 2, [6], pp. 418-419).

If the translation complement  $C$  acts transitively on the 1-dimensional subspaces of  $S$ , then we have a family of translation planes  $\mathcal{A}_{k,w,z,p,q}$  such that the normalized spreads belonging to these planes have the form

$$\begin{aligned} &\{S\} \cup \left\{ \beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \beta \in \mathbb{R} \right\} \cup \\ &\left\{ \beta \begin{pmatrix} u - (w+1) \sin(l) \cos(l) + z \sin^2(l) & w \cos^2(l) - z \sin(l) \cos(l) - \sin^2(l) \\ \cos^2(l) - z \sin(l) \cos(l) - w \sin^2(l) & z \cos^2(l) + (w+1) \sin(l) \cos(l) + u \end{pmatrix}, u \in \mathbb{R}, \beta > 0 \right\} \cup \\ &\left\{ \beta \begin{pmatrix} (p-1) \sin(l_1) \cos(l_1) - q \sin^2(l_1) - u & q \sin(l_1) \cos(l_1) - p \cos^2(l_1) - \sin^2(l_1) \\ \cos^2(l_1) + q \sin(l_1) \cos(l_1) + p \sin^2(l_1) & (1-p) \sin(l_1) \cos(l_1) - q \cos^2(l_1) - u \end{pmatrix}, u \in \mathbb{R}, \beta < 0 \right\}, \quad (27) \end{aligned}$$

where  $l = \frac{1}{k} \ln \beta$ ,  $l_1 = \frac{1}{k} \ln(-\beta)$  (cf. [15], Proposition 4.1, p. 6, and [6], Satz 3, pp. 422-423). The spreads (25), respectively (26), respectively (27) coincide with the image of the section  $\sigma$  in (2) such that the well defined functions  $a(r, t)$  and  $b(r, t)$  are given in assertion a), respectively b), respectively c).

Since in all three cases we have  $a(r, k\pi) = 1$ ,  $b(r, k\pi) = 0$ ,  $r > 0$ ,  $k \in \{0, 1\}$ , Remark 1 shows that  $\mathcal{K}$  has the form as in the assertion. In case a), respectively b) for  $\beta > 0$  one gets  $a(r, \frac{\pi}{4} + \pi) = \frac{\sqrt{2}\beta}{\sqrt{\beta^2 - z\beta\frac{2+s}{1+s} - w\beta\frac{2}{1+s}}}$ , respectively

$a(r, \frac{\pi}{4} + \pi) = \sqrt{\frac{2}{1-z-w-2\ln\beta+z\ln\beta+(\ln\beta)^2}}$ . In case c) for  $u = 0$ ,  $\beta > 0$  we get that  $a(1, t_j)$  is constant. These relations give a contradiction to the condition  $a(r, t + k\pi) = a(1, t)$ ,  $r > 0$ ,  $t \in [0, 2\pi)$ ,  $k \in \{0, 1\}$  of Proposition 11. Hence in all cases  $Q^*$  is not decomposable and quasi-simple (cf. Theorem 12).  $\square$

**Proposition 19.** *Let  $Q$  be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane  $\mathcal{A}$  such that the translation complement  $C$  of the 7-dimensional collineation group  $\Gamma$  of  $\mathcal{A}$  has an orbit of dimension 1 on  $W \setminus \{0\}$ ,  $C$  leaves only  $S$  in the set of lines through the origin fixed and the kernel of its action on  $S$  is zero-dimensional. Then the set  $\Lambda_{Q^*}$  of all left translations of the multiplicative loop  $Q^*$  is given by the range of the section (2) defined as follows: For  $\alpha \geq -\frac{\beta^2}{2}$  one has*

$$a(r, t) = \sqrt{\frac{\alpha^2 + \beta^2}{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} + \left(\alpha + \frac{\beta^2}{2}\right) \left(\alpha + \frac{q-1}{q}\beta^2\right) - \frac{p\beta}{q} \left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}}}},$$

$$b(r, t) = \frac{\frac{p}{q}\alpha \left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}} - \frac{p}{q}(\alpha^2 + \beta^2) + \frac{1-q}{q}\beta\alpha^2 + \frac{\alpha\beta^3}{6q} - \frac{\beta^3\alpha}{2} + \frac{\beta^3}{2q} + \frac{\beta^3}{2} + \alpha\beta}{\sqrt{\alpha^2 + \beta^2} \sqrt{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} + \left(\alpha + \frac{\beta^2}{2}\right) \left(\alpha + \frac{q-1}{q}\beta^2\right) - \frac{p\beta}{q} \left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}}}},$$

with

$$r \cos(t) = \alpha \sqrt{\frac{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} + \left(\alpha + \frac{\beta^2}{2}\right) \left(\alpha + \frac{q-1}{q}\beta^2\right) - \frac{p\beta}{q} \left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}}}{\alpha^2 + \beta^2}},$$

$$r \sin(t) = -\beta \sqrt{\frac{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} + \left(\alpha + \frac{\beta^2}{2}\right) \left(\alpha + \frac{q-1}{q}\beta^2\right) - \frac{p\beta}{q} \left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}}}{\alpha^2 + \beta^2}}.$$

For  $\alpha < -\frac{\beta^2}{2}$  we get

$$a(r, t) = \sqrt{\frac{\alpha^2 + \beta^2}{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} - \left(\alpha + \frac{\beta^2}{2}\right) \left(\frac{\alpha z}{q} + \frac{(z+1)\beta^2}{q}\right) - \frac{w\beta}{q} \left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}}}},$$

$$b(r, t) = \frac{\frac{w}{q}\alpha \left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}} + \frac{p}{q}(-\alpha^2 - \beta^2) + \left(\frac{z+1}{q}\alpha\beta - \frac{z\beta}{q}\right) \left(\alpha + \frac{\beta^2}{2}\right) - \frac{\alpha\beta^3}{3q} + \frac{\beta^3}{2q}}{\sqrt{\alpha^2 + \beta^2} \sqrt{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} - \left(\alpha + \frac{\beta^2}{2}\right) \left(\frac{\alpha z}{q} + \frac{(z+1)\beta^2}{q}\right) - \frac{w\beta}{q} \left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}}}},$$

with

$$r \cos(t) = \alpha \sqrt{\frac{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} - \left(\alpha + \frac{\beta^2}{2}\right) \left(\frac{\alpha z}{q} + \frac{(z+1)\beta^2}{q}\right) - \frac{w\beta}{q} \left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}}}{\alpha^2 + \beta^2}},$$

$$r \sin(t) = -\beta \sqrt{\frac{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} - \left(\alpha + \frac{\beta^2}{2}\right) \left(\frac{\alpha z}{q} + \frac{(z+1)\beta^2}{q}\right) - \frac{w\beta}{q} \left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}}}{\alpha^2 + \beta^2}}.$$

The quasifields  $Q_{w,z,p,q}$  coordinatize a family of planes  $\mathcal{A}_{w,z,p,q}$  such that the parameters  $w, z, p, q$  satisfy  $(3w)^2 \leq -16z(z+1)$ ,  $(3p)^2 \leq 16q(q-1)$ ,  $q > 0$ ,  $z < 0$  and  $(w, z, p, q) \neq (0, -\frac{1}{3}, 0, 3)$ .

The multiplicative loops  $Q_{w,z,p,q}^*$  of the quasifields  $Q_{w,z,p,q}$  are not decomposable and quasi-simple. The left translations of  $Q_{w,z,p,q}^*$  corresponding to the kernel of  $Q_{w,z,p,q}$  have the form  $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ ,  $0 \neq r \in \mathbb{R}$ , if and only if  $w = p = 0$ ,  $q = -z = 1$ .

*Proof.* By Satz 5 in [6], the planes  $\mathcal{A}_{w,z,p,q}$  are determined by the normalized spreads which have the form

$$\{S\} \cup \left\{ \begin{pmatrix} \alpha & -\frac{p}{q}\alpha + \frac{p}{q}\left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}} + \frac{(1-q)\beta}{q}\left(\alpha + \frac{\beta^2}{2}\right) - \frac{\beta^3}{3q} \\ \beta & -\frac{p}{q}\beta + \frac{\beta^2}{2q} + \left(\alpha + \frac{\beta^2}{2}\right) \end{pmatrix}, \beta \in \mathbb{R}, \alpha \geq -\frac{\beta^2}{2} \right\} \cup \\ \left\{ \begin{pmatrix} \alpha & -\frac{p}{q}\alpha + \frac{w}{q}\left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}} + \frac{(z+1)\beta}{q}\left(\alpha + \frac{\beta^2}{2}\right) - \frac{\beta^3}{3q} \\ \beta & -\frac{p}{q}\beta + \frac{\beta^2}{2q} - \frac{z}{q}\left(\alpha + \frac{\beta^2}{2}\right) \end{pmatrix}, \beta \in \mathbb{R}, \alpha < -\frac{\beta^2}{2} \right\}.$$

These spreads coincide with the image of the section  $\sigma$  in (2) such that the well defined functions  $a(r, t)$  and  $b(r, t)$  are given in the assertion. One gets that  $a(r, 0) = 1$  and  $a(r, \pi) = \sqrt{\frac{-q}{z}}$  for all  $r > 0$ .

For  $\beta > 2$  we obtain

$$a\left(r, \frac{\pi}{4} + \pi\right) = \frac{\sqrt{2}\beta}{\sqrt{\frac{\beta^4}{3q} - \frac{\beta^3}{2q} - \frac{w}{q}\beta\left(\beta - \frac{\beta^2}{2}\right)^{\frac{3}{2}} + \left(\beta - \frac{\beta^2}{2}\right)\left(\frac{z+1}{q}\beta^2 - \frac{\beta z}{q}\right)}}.$$

The loop  $Q_{w,z,p,q}^*$  is not decomposable since we have a contradiction to the condition  $a\left(r, \frac{\pi}{4} + k\pi\right) = a\left(1, \frac{\pi}{4}\right)a(r, k\pi)$ ,  $r > 0$ ,  $k \in \{0, 1\}$  (cf. Proposition 11). Hence  $Q_{w,z,p,q}^*$  is quasi-simple (cf. Theorem 12). As  $a(r, k\pi) = 1$  and  $b(r, k\pi) = 0$ ,  $r > 0$ ,  $k \in \{0, 1\}$  holds precisely if  $w = p = 0$ ,  $q = -z = 1$  the last assertion follows.  $\square$

**Proposition 20.** *Let  $Q$  be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane  $\mathcal{A}$  such that the translation complement  $C$  of the 7-dimensional collineation group  $\mathcal{A}$  fixes two distinct lines  $\{S, W\}$  through the origin and leaves on  $S$  one or two 1-dimensional subspaces invariant. Then the multiplicative loop  $Q^*$  can be given by one of the following sets  $\Lambda_{Q^*}$  of the left translations of  $Q^*$  having the form (20):*

a)

$$a(r, 0) = r^{\frac{1-w}{1+w}}, \quad b(r, 0) = c\left(r^{\frac{w-1}{w+1}} - r^{\frac{1-w}{1+w}}\right),$$

with  $r = s^{\frac{w+1}{2}}$ ,  $s > 0$ ,  $t = -\varphi$ , where  $s$  and  $\varphi$  are variables of the spreads (28). The quasifields  $Q_{w,c}$  coordinatize a family of planes  $\mathcal{A}_{w,c}$  such that for the parameters  $w \neq 1, c$  one has  $0 < w$  and  $(w-1)^2 c^2 \leq 4w$ .



b)

$$a(r, 0) = 1, \quad b(r, 0) = \frac{\ln r}{d},$$

with  $r = e^s$ ,  $t = -\varphi$ , where  $s$  and  $\varphi$  are variables of the spreads (29). The quasifields  $Q_d$  coordinatize a one-parameter family of planes  $\mathcal{A}_d$  such that  $4d^2 \geq 1$ .

In both cases  $Q^*$  is decomposable and contains the group  $\text{SO}_2(\mathbb{R})$ .

*Proof.* If the group  $C$  fixes two 1-dimensional subspaces of  $S$ , respectively only one 1-dimensional subspace of  $S$ , then one obtains a family of translation planes corresponding to the normalized spreads

$$\{S, W\} \cup \left\{ \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} s & c(s^w - s) \\ 0 & s^w \end{pmatrix}, s, \varphi \in \mathbb{R}, s > 0 \right\} \quad (28)$$

(cf. [7], Satz 1 and [9], p. 15), respectively

$$\{S, W\} \cup \left\{ \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} e^s & e^{s \frac{s}{d}} \\ 0 & e^s \end{pmatrix}, s, \varphi \in \mathbb{R} \right\}, \quad (29)$$

(cf. [7], Satz 2 and [9], p. 15). In both cases these spreads coincide with the set  $\Lambda = \text{SO}_2(\mathbb{R})\mathcal{K}$  given in (20) such that the set  $\mathcal{K}$  corresponding to the kernel  $K_r$  of  $Q$  is determined by the functions  $a(r, 0)$ ,  $b(r, 0)$  as in assertion a), respectively b).  $\square$

**Remark 21.** In [2] D. Betten constructed 4-dimensional locally compact non-desarguesian planes  $\mathcal{A}_f$  corresponding to continuous, non-linear, strictly monotone functions  $f$  defined for  $0 \leq u \in \mathbb{R}$  with  $f(0) = 0$  and  $\lim_{u \rightarrow \infty} f(u) = \infty$ . The planes  $\mathcal{A}_f$  are determined by the normalized spreads

$$\left\{ \begin{pmatrix} u \cos \varphi & -\frac{f(u) \sin \varphi}{f(1)} \\ u \sin \varphi & \frac{f(u) \cos \varphi}{f(1)} \end{pmatrix}, u > 0, \varphi \in [0, 2\pi) \right\}.$$

These spreads coincide with the set  $\Lambda = \text{SO}_2(\mathbb{R})\mathcal{K}$  given in (20) such that the set  $\mathcal{K}$  corresponding to the kernel  $K_r$  of the quasifield  $Q_f$  coordinatizing  $\mathcal{A}_f$  is determined by the functions  $a(r, 0) = \sqrt{\frac{uf(1)}{f(u)}}$ ,  $b(r, 0) = 0$  with  $r = \sqrt{\frac{uf(u)}{f(1)}}$ ,  $t = -\varphi$ ,  $u \neq 0$ . For  $f(u) = f(1)u^w$  these planes are planes in Proposition 20 a) with  $c = 0$  and  $a(r, 0) = r^{\frac{1-w}{1+w}}$ . Otherwise the full collineation group of the planes  $\mathcal{A}_f$  has dimension 6.

**Proposition 22.** Let  $Q$  be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane  $\mathcal{A}$  such that the translation

complement  $C$  of the 7-dimensional collineation group of  $\mathcal{A}$  fixes two distinct lines  $\{S, W\}$  through the origin and acts transitively on the spaces  $P_S$  and  $P_W$  of all 1-dimensional subspaces of  $S$ , respectively  $W$ . Then the multiplicative loop  $Q^*$  of  $Q$  can be given by one of the following sets  $\Lambda_{Q^*}$  of the left translations of  $Q^*$ :

a)  $\Lambda_{Q^*}$  is the range of the section (2) with

$$\begin{aligned} a(r, u) &= \sqrt{\frac{dD}{de^{2(qt-ps)} + de^{2q\pi} + e^{qt-ps+q\pi} (2d \cos s \cos t + (c^2 + 1 + d^2) \sin s \sin t)}}, \\ b(r, u) &= \frac{e^{2(qt-ps)} [(-c^2 - 1 + d^2) \cos t \sin t - c(c^2 + 1 + d^2) \sin^2 t]}{\sqrt{dD [d(e^{2(qt-ps)} + e^{2q\pi}) + e^{qt-ps+q\pi} (2d \cos s \cos t + (d^2 + c^2 + 1) \sin s \sin t)]}} + \\ &+ \frac{e^{qt-ps+q\pi} (\cos s \cos t + d \sin s \sin t + c \cos s \sin t)}{\sqrt{dD [d(e^{2(qt-ps)} + e^{2q\pi}) + e^{qt-ps+q\pi} (2d \cos s \cos t + (d^2 + c^2 + 1) \sin s \sin t)]}}, \end{aligned}$$

such that

$$\begin{aligned} r \cos u &= \frac{e^{qt-ps} (\cos s \cos t + c \sin t \cos s + d \sin t \sin s) + e^{q\pi}}{1 + e^{q\pi}} a^{-1}(r, u), \\ r \sin u &= -\frac{e^{qt-ps} (d \cos s \sin t - \sin s \cos t - c \sin s \sin t)}{1 + e^{q\pi}} a^{-1}(r, u), \\ D &= e^{2(qt-ps)} ((\cos t + c \sin t)^2 + d^2 \sin^2 t) + e^{2q\pi} + 2e^{qt-ps+q\pi} (\cos s \cos t + c \cos s \sin t + d \sin s \sin t). \end{aligned}$$

The quasifields  $Q_{p,q,c,d}$  coordinatize a family of planes  $\mathcal{A}_{p,q,c,d}$  such that the parameters  $p, q, c, d$  satisfy the conditions

$$\begin{aligned} p &= q > 0 & \text{and} & \quad -1 \leq d < 0, \\ q > 0, p &= \frac{k-1}{k+1}q, k = 1, 2, 3, \dots & \text{and} & \quad d > 0, \end{aligned}$$

$$-(q+p)^2 A + (q-p)^2 B - 4AB \geq 0, \text{ where } A = \frac{(d-1)^2 + c^2}{4d} \text{ and } B = \frac{(d+1)^2 + c^2}{4d}.$$

The multiplicative loops  $Q^*$  of the quasifields  $Q_{p,q,c,d}$  are not decomposable and quasi-simple.

b)  $\Lambda_{Q^*}$  has the form (15) with

$$a(1, u) = \sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}, \quad b(1, u) = \frac{\sin nt \cos nt (d^2 - 1 - c^2) - c \sin^2 nt (d^2 + 1 + c^2)}{d \sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}}$$

such that

$$r \cos u = \frac{s(\cos nt \cos mt + c \sin nt \cos mt + d \sin nt \sin mt)}{\sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}}, \quad r \sin u = \frac{s(d \sin nt \cos mt - \cos nt \sin mt - c \sin nt \sin mt)}{\sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}}$$

and  $s \geq 0$ .

The quasifields  $Q_{m,n,c,d}$  coordinatize a family of planes  $\mathcal{A}_{m,n,c,d}$  such that the parameters  $m, n \in \mathbb{Z}$ ,  $(m, n) = 1$ ,  $c, d \in \mathbb{R}$  satisfy the conditions

$$\begin{aligned} m &= n = 1 & \text{and} & \quad -1 \leq d < 0 \\ m &= 1, 2, 3, \dots & n &= m + 1 & \text{and} & \quad d > 0 \\ m &= 1, 3, 5, \dots & n &= m + 2 & \text{and} & \quad d > 0 \end{aligned}$$

$$(n-m)^2 B \geq (n+m)^2 A, \text{ where } A = \frac{(d-1)^2 + c^2}{4d} \text{ and } B = \frac{(d+1)^2 + c^2}{4d}.$$

The loops  $Q_{m,n,c,d}^*$  are split extensions of the normal subgroup  $\widetilde{N}^* \cong \mathbb{R}$  corresponding to the connected component of  $\left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, 0 \neq u \in \mathbb{R} \right\}$  with a subloop homeomorphic to the 1-sphere.

*Proof.* If the translation complement  $C$  acts transitively on the product space  $P_S \times P_w$ , then there is a family of translation planes corresponding to the normalized spreads

$$\{S, W\} \cup \left\{ \left( \begin{array}{cc} \frac{\alpha(s,t)+e^{q\pi}}{1+e^{q\pi}} & \frac{\gamma(s,t)-c\alpha(s,t)}{d(1+e^{q\pi})} \\ \frac{\beta(s,t)}{1+e^{q\pi}} & \frac{\delta(s,t)-c\beta(s,t)+de^{q\pi}}{d(1+e^{q\pi})} \end{array} \right), s, t \in \mathbb{R} \right\}$$

such that  $\alpha(s, t) = e^{qt-ps}(\cos s \cos t + c \sin t \cos s + d \sin t \sin s)$ ,  
 $\beta(s, t) = e^{qt-ps}(d \cos s \sin t - \sin s \cos t - c \sin s \sin t)$ ,  
 $\gamma(s, t) = e^{qt-ps}(d \cos t \sin s - \sin t \cos s + c \cos t \cos s)$ ,  
 $\delta(s, t) = e^{qt-ps}(d \cos t \cos s + \sin t \sin s - c \cos t \sin s)$  (cf. [7], Satz 3, pp. 135-136). These spreads coincide with the image of the section  $\sigma$  in (2) with the well defined functions  $a(r, u)$  and  $b(r, u)$  as in assertion a). For  $s = 0$  we get a contradiction to the condition  $a(r_j, u_j) = a(r_j, 0)a(1, u_j)$  which must hold for decomposable  $Q^*$ . It follows that  $Q^*$  is not decomposable and hence quasi-simple (cf. Theorem 12).

If the translation complement  $C$  does not act transitively on the product space  $P_S \times P_W$ , then there is a family of translation planes which correspond to the normalized spreads

$$\{S, W\} \cup \left\{ \left( \begin{array}{cc} s & 0 \\ 0 & s \end{array} \right) \left( \begin{array}{cc} a_{11}(t) & -\frac{c}{d}a_{11}(t) + \frac{1}{d}a_{21}(t) \\ a_{12}(t) & -\frac{c}{d}a_{12}(t) + \frac{1}{d}a_{22}(t) \end{array} \right), s \geq 0, t \in \mathbb{R} \right\}$$

with  $a_{11}(t) = \cos nt \cos mt + c \sin nt \cos mt + d \sin nt \sin mt$ ,  
 $a_{12}(t) = d \sin nt \cos mt - \cos nt \sin mt - c \sin nt \sin mt$ ,  
 $a_{21}(t) = d \cos nt \sin mt - \sin nt \cos mt + c \cos nt \sin mt$ ,  
 $a_{22}(t) = d \cos nt \cos mt + \sin nt \sin mt - c \cos nt \sin mt$  (cf. [7], Satz 4, pp. 142-144). These spreads coincide with the set  $\Lambda_{Q^*}$  in (15) such that the periodic functions  $a(1, t)$  and  $b(1, t)$  are given in assertion b). As in the proof of Proposition 16 a) it follows that the loop  $Q_{m,n,c,d}^*$  is a split extension as in the assertion.  $\square$

**Corollary 23.** *Let  $\mathcal{A}$  be a 4-dimensional locally compact non-desarguesian topological plane which admits an at least 7-dimensional collineation group  $\Gamma$ . If the quasifield  $Q$  coordinatizing  $\mathcal{A}$  is constructed with respect to two lines such that their intersection points with the line at infinity are contained in the 1-dimensional orbit of  $\Gamma$  or contain the set of the fixed points of  $\Gamma$ , then for the multiplicative loop  $Q^*$  of  $Q$  one of the following holds:*

- a)  $Q^*$  is quasi-simple and not decomposable. Such quasifields  $Q$  are described by Propositions 16 b), 16 c), 17), 18), 19) and in Proposition 22 a).
- b)  $Q^*$  is quasi-simple but decomposable and it is a product  $SO_2(\mathbb{R})B$ , where  $B$  is a 1-dimensional loop homeomorphic to  $\mathbb{R}$ . The quasifields  $Q$  of this type

are described in Proposition 20.

c)  $Q^*$  is a split extension of the group  $\widetilde{N}^* \cong \mathbb{R}$  with a loop homeomorphic to the 1-sphere. The quasifields of this type are described in Propositions 16 a) and 22 b).

*Proof.* A locally compact topological quasifield coordinatizing the translation plane  $\mathcal{A}$  and constructed with respect to two lines satisfying the assumptions is isotopic to a quasifield given in Betten's classification (cf. [11], p. 321, [3] Satz 5). For isotopic loops  $Q_1^*$  and  $Q_2^*$  the following holds: The group generated by their left translations, every subgroup and all nuclei of them are isomorphic (cf. [14], Lemmata 1.9, 1.10, p. 20). From these facts we get: If  $Q_1$  is quasisimple and not decomposable, then also  $Q_2$  is quasisimple and not decomposable. If  $Q_1$  contains the subgroup  $SO_2(\mathbb{R})$ , then also  $Q_2$  contains the group  $SO_2(\mathbb{R})$ . If  $Q_1$  is a split extension of  $\widetilde{N}^*$  with a 1-dimensional compact loop, then the same holds for  $Q_2$ .  $\square$

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