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Linearised Symmetry Condition for Systems of SODEs

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SEVENTH FRAMEWORK PROGRAMME

Abstract

The aim of this talk is

- to present infinitesimal symmetries for a system of SODEs,
- introduce the linearised symmetry condition,
- give a coordinate free characterization of the infinitesimal symmetries in terms of the Lie bracket of vector fields
- present new results about (first and higher order) differential equations associated to given Lie groups or Lie algebras;

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System of second order ordinary differential equations

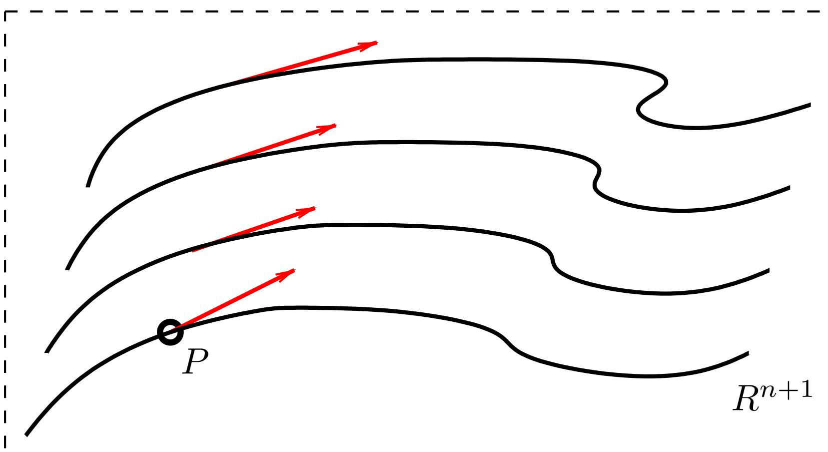
Coordinates in $\mathbb{R}^n$: $y = (y^1, \ldots, y^n)$,

A system of second order ordinary differential equations in $\mathbb{R}^n$:

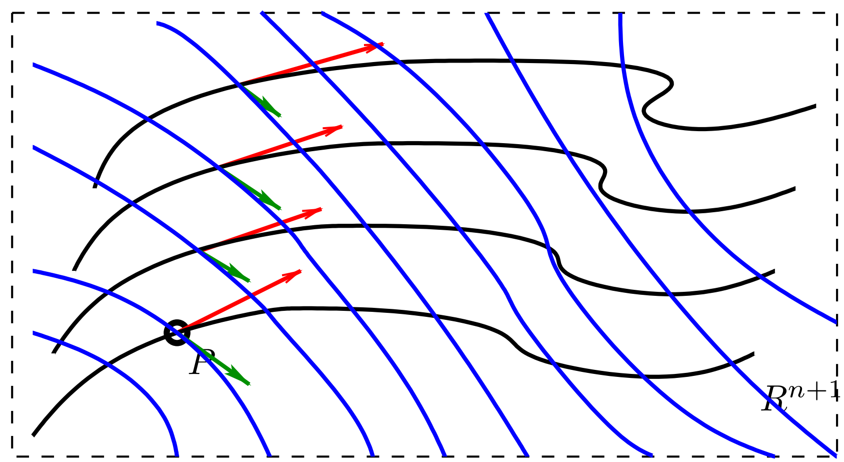
$$\frac{d^2y^1}{dx^2} = \omega^1(x, y, \dot{y}), \qquad \ldots \qquad , \frac{d^2y^n}{dx^2} = \omega^n(x, y, \dot{y}).$$

First and second order ODEs:

First order ODE:



Second order ODE:



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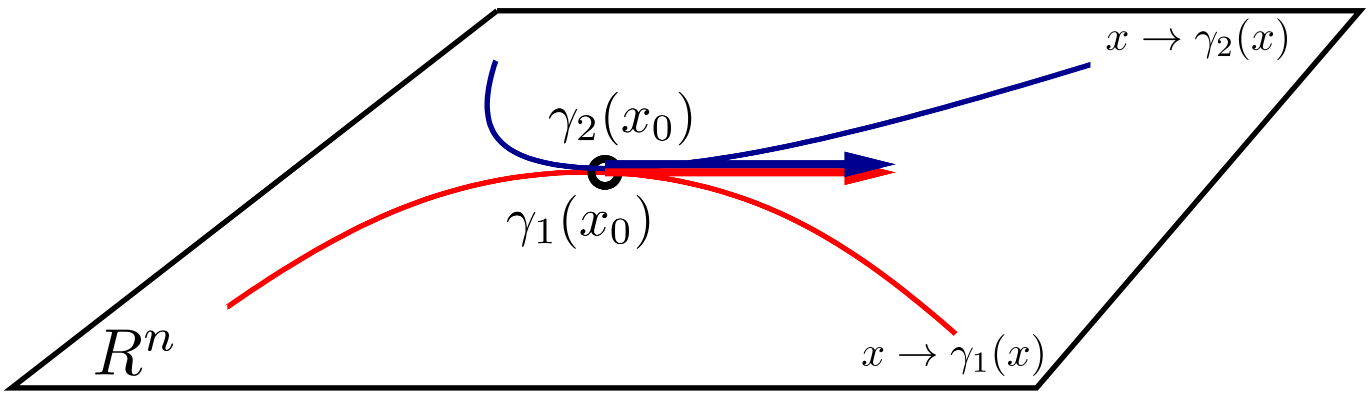
Second order jet space

The curves γ_1 and γ_2 are equivalent up to order 2 at x_0 if

$$\gamma_1(x_0) = \gamma_2(x_0), \qquad \frac{d\gamma_1}{dx}\bigg|_{x_0} = \frac{d\gamma_2}{dx}\bigg|_{x_0}, \qquad \frac{d^2\gamma_1}{dx^2}\bigg|_{x_0} = \frac{d^2\gamma_2}{dx^2}\bigg|_{x_0}.$$



$$j_{2,x_0}(\gamma) \in J_2(\mathbb{R}, \mathbb{R}^n)$$

Coordinates for $J_2(\mathbb{R}, \mathbb{R}^n)$: $\quad j_{2,x_0}(\gamma) \simeq (x_0, y^i, \dot{y}^i, \ddot{y}^i)$

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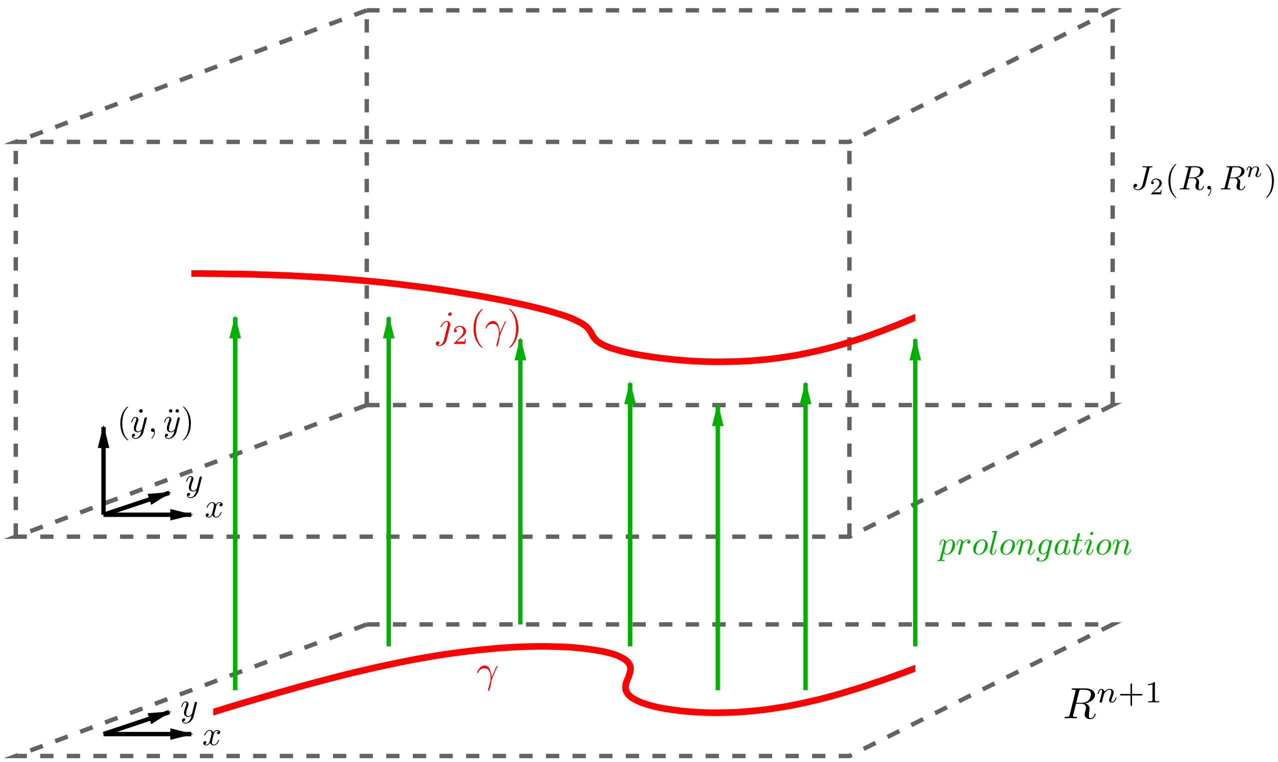
Prolongation of a curve

$$\gamma : x \longrightarrow \big(x, y^i(x)\big) \qquad\qquad j_2(\gamma): x \longrightarrow \big(x, y^i(x), \dot{y}^i(x), \ddot{y}^i(x)\big)$$



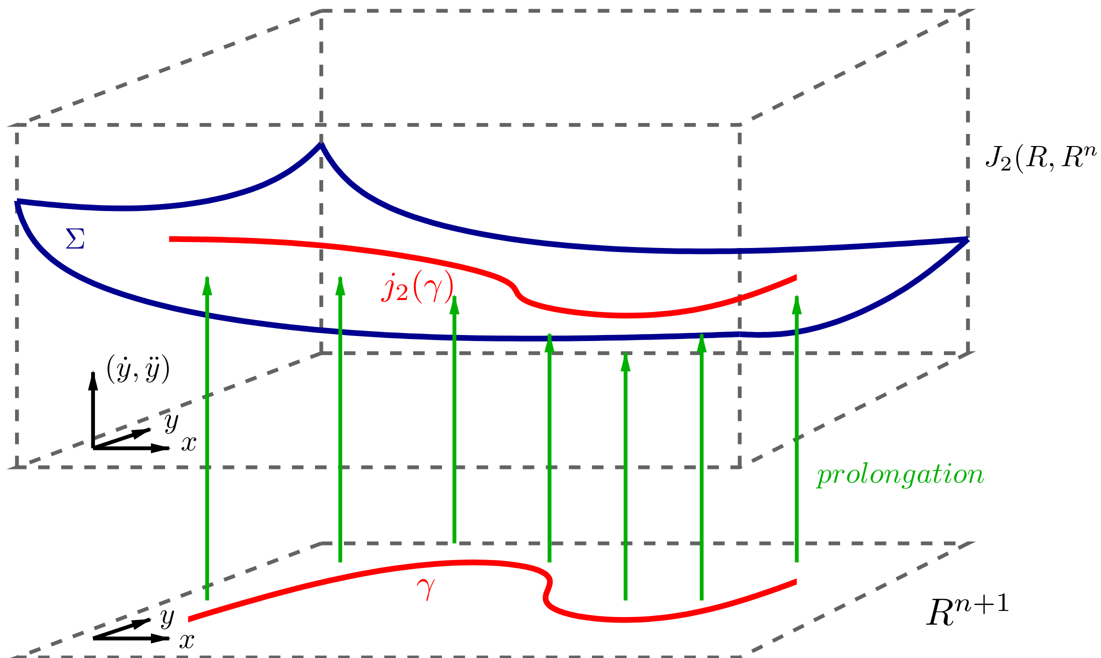
Second order ODE: submanifold of the 2-jet space

Second order system:

- $\ddot{y}^j = \omega^j(x, y, \dot{y})$
- $\Sigma^j(x, y, \dot{y}, \ddot{y}) := \ddot{y}^j - \omega^j(x, y, \dot{y}) \quad \Rightarrow \quad \Sigma^j(x, y, \dot{y}, \ddot{y}) \equiv 0,$

Solution:

- $\dfrac{d^2 y^j}{dx^2}(x) = \omega^j(x, y(x), \dot{y}(x)),$
- $\Sigma^j(x, y^i(x), \frac{dy^i}{dx}(x), \frac{d^2 y^i}{dx^2}(x)) = 0,$
- $j_2(\gamma) \subset \Sigma.$



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The prolongation of a diffeomorphism

$$\begin{array}{ccc} J_2(\mathbb{R},\mathbb{R}^n) & \xrightarrow{S^2} & J_2(\mathbb{R},\mathbb{R}^n) \\ \text{prolongation}\uparrow & & \uparrow\text{prolongation} \\ \mathbb{R}\times\mathbb{R}^n & \xrightarrow{\quad S \quad} & \mathbb{R}\times\mathbb{R}^n \end{array}$$

- Diffeomorphism:

$$\begin{array}{ccl} \mathbb{R}\times\mathbb{R}^n & \xrightarrow{S} & \mathbb{R}\times\mathbb{R}^n \\ (x,y^i) & & (\bar{x},\bar{y}^i) = \big(\varphi(x,y),\psi^i(x,y)\big) \end{array}$$

- Associated map:

$$\begin{array}{ccl} J_2(\mathbb{R},\mathbb{R}^n) & \xrightarrow{S^2} & J_2(\mathbb{R},\mathbb{R}^n) \\ (x,y^i,\dot{y}^i,\ddot{y}^i) & & (\bar{x},\bar{y}^i,\dot{\bar{y}}^i,\ddot{\bar{y}}^i) = \left(\varphi,\psi^i,\frac{D\psi^i}{D\varphi},\frac{D^2\psi^i\cdot D\varphi - D\psi^i\cdot D^2\varphi}{(D\varphi)^3}\right) \end{array}$$

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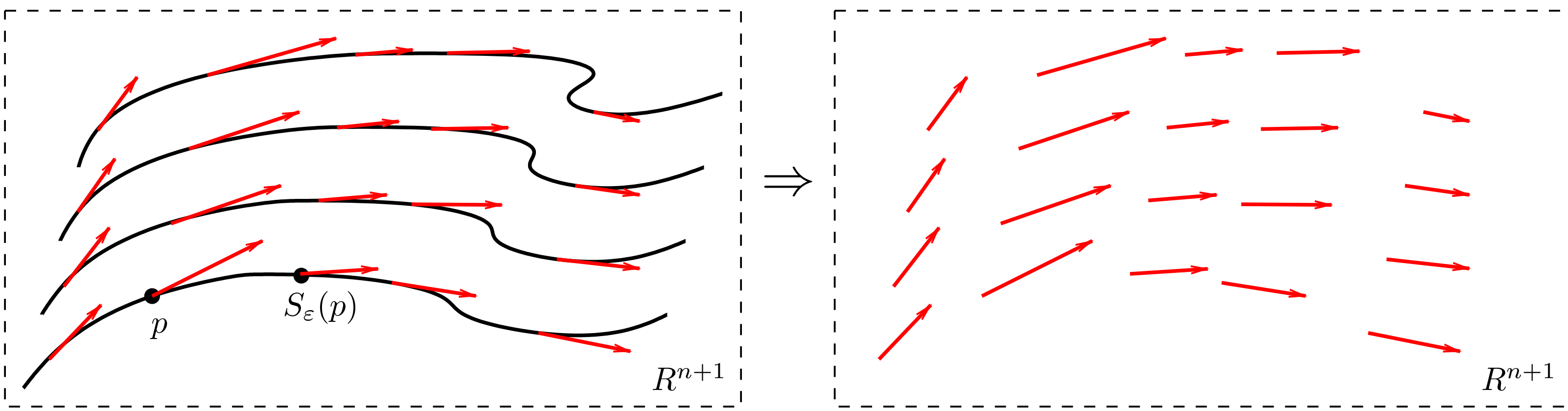
Infinitesimal generators

1-parameter group of diffeomorphism:

$$\{S_\varepsilon\}_{\varepsilon\in\mathbb{R}} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \qquad S_\varepsilon = (\varphi_\varepsilon, \psi^i_\varepsilon)$$

Infinitesimal generator:



$$S_\varepsilon = (\varphi_\varepsilon, \psi^i_\varepsilon) \qquad \Rightarrow \qquad Y = \xi \frac{\partial}{\partial x} + \eta^i \frac{\partial}{\partial y^i}$$

$$S^2_\varepsilon = \left(\varphi_\varepsilon, \psi^i_\varepsilon, \frac{D\psi^i_\varepsilon}{D\varphi_\varepsilon}, \frac{D^2\psi^i_\varepsilon \cdot D\varphi_\varepsilon - D\psi^i_\varepsilon \cdot D^2\varphi_\varepsilon}{(D\varphi_\varepsilon)^3}\right) \qquad \Rightarrow \qquad Y^2 = \xi \frac{\partial}{\partial x} + \eta^i \frac{\partial}{\partial y^i} + \eta^i_1 \frac{\partial}{\partial \dot{y}^i} + \eta^i_2 \frac{\partial}{\partial \ddot{y}^i}$$

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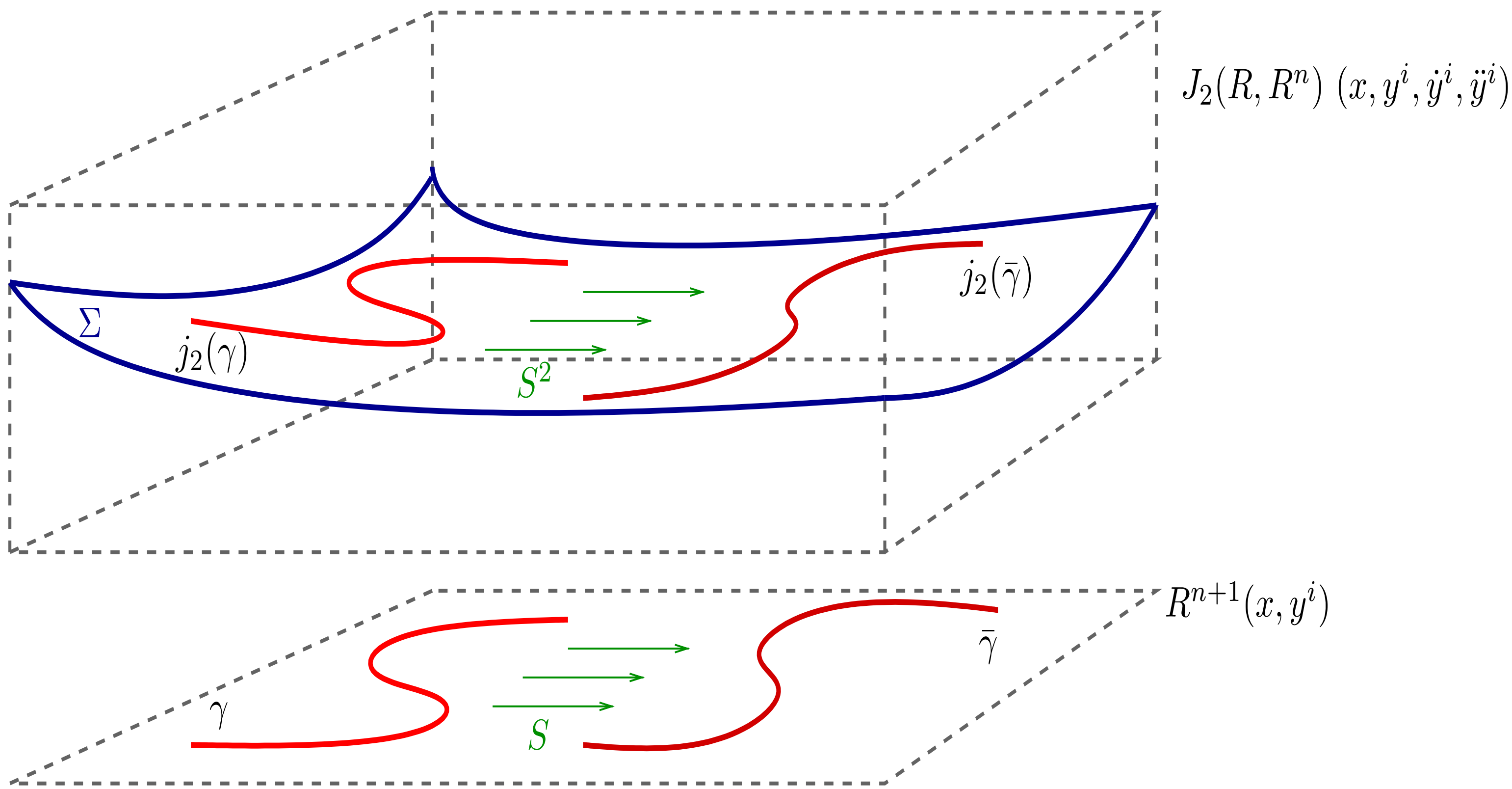
Symmetry

Definition: $S : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a symmetry, if S transforms solutions into solutions:

$$\Sigma^j(x, y, \dot{y}, \ddot{y}) = 0 \quad \Rightarrow \quad \Sigma^j(\bar{x}, \bar{y}, \dot{\bar{y}}, \ddot{\bar{y}}) = 0,$$

$$j_2(\gamma) \subset \Sigma \quad \Rightarrow \quad j_2(\bar{\gamma}) \subset \Sigma$$



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Linearised symmetry condition

Symmetry $\Rightarrow$ S_ε $\Leftrightarrow$ $Y = \xi\partial_x + \eta^i\partial_{y^i}$

$\Rightarrow$ S_ε^2 : prolongation of S_ε $\Rightarrow$ transformation of Σ

$\Rightarrow$ Y^2 : infinitesimal generator $\Rightarrow$ tangent to Σ

$\Rightarrow$ LSC : PDE on Y

S_ε $\Rightarrow$ $S_\varepsilon = (\bar{x}_\varepsilon, \bar{y}^i_\varepsilon)$

$\Rightarrow$ $S^2_\varepsilon = (\bar{x}_\varepsilon, \bar{y}^i_\varepsilon, \dot{\bar{y}}^i_\varepsilon, \ddot{\bar{y}}^i_\varepsilon)$ $\Rightarrow$ $Y^2 = \frac{dS^2_\varepsilon}{d\varepsilon}$ $\Rightarrow$ $Y^2(\Sigma^j) \equiv 0$

$$\Rightarrow \left(D^2\eta^j - 2\omega^j D\xi - \dot{y}^j D^2\xi\right) - \xi\frac{\partial\omega^j}{\partial x} - \eta^i\frac{\partial\omega^j}{\partial y^i} - \left(D\eta^i - \dot{y}^i D\xi\right)\frac{\partial\omega^j}{\partial \dot{y}^i} \equiv 0$$

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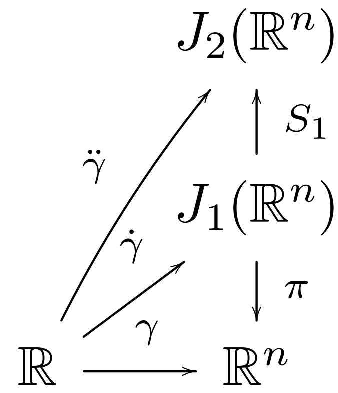
Lie bracket condition and the LSC



Spray: $\mathcal{S}_1 = \frac{\partial}{\partial x} + \dot y^i \frac{\partial}{\partial y^i} + \omega^i(x, y, \dot y)\frac{\partial}{\partial \dot y^i}$

Curve of the spray: $\gamma : \mathbb{R} \to \mathbb{R}^n, \quad \mathcal{S}_1(\dot\gamma) = \ddot\gamma$

Proposition: The vector field $X = \xi \frac{\partial}{\partial x} + \eta^i \frac{\partial}{\partial y^i}$ is an infinitesimal symmetry if and only if

$$[\mathcal{S}_1, \bar X_1] \equiv 0.$$

where $\bar X_1 = X_1 - \xi S_1$.

Corollary: A time-preserving infinitesimal transformation $(\xi = 0)$ is an infinitesimal symmetry if and only if

$$[\mathcal{S}_1, X_1] \equiv 0.$$

Differential equation associated to Lie groups

G Lie group, $\mathfrak{g}$ Lie algebra $\overset{?}{\Rightarrow}$ ODE $(\Sigma = 0)$

Prop: $\Sigma = 0$ is a kth order ODE and $X \in \mathfrak{g}$, then $X^{(k)}(\Sigma) \equiv 0$.

$$\mathfrak{g} \supset \{X_1, \dots, X_n\} \overset{prolong}{\Rightarrow} \{X_i^{(k)}\}_{i=1\dots n} \overset{k=n-2}{\Rightarrow} \det(X_i^{(n-2)})_{i=1\dots n} = 0$$

Olver: Classification of Lie algebras of vector fields in the real plane

Locally primitive algebras	
$\mathbb{R} \ltimes \mathbb{R}^2$	$1 + \dot{y}^2 = 0$
$\mathfrak{sl}(2)$	$1 + \dot{y}^2 = 0$
$\mathfrak{so}(3)$	$1 + \dot{y}^2 = 0$
$\mathbb{R}^2 \ltimes \mathbb{R}^2$	$\ddot{y} = 0,\ \dot{y} = -1$
$\mathfrak{sl}(2) \ltimes \mathbb{R}^2$	$\ddot{y} = 0$
$\mathfrak{gl}(2) \ltimes \mathbb{R}^2$	$\ddot{y} = 0,\ 3y^{(4)}\ddot{y} - 5(y^{(3)})^2 = 0$
$\mathfrak{so}(3,1)$	$y^{(3)}(1+\dot{y}^2) - 3\dot{y}\ddot{y}^2 = 0$
$\mathfrak{sl}(3) \ltimes \mathbb{R}^2$	$\ddot{y} = 0,\ 9\ddot{y}^2 y^{(5)} - 45\ddot{y}y^{(3)}y^{(4)} + 40(y^{(3)})^3 = 0$

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Imprimitive algebras:

$\mathfrak{sl}(2)$	$0=0$
$\mathbb{R}\ltimes\mathbb{R}^2$	$\dot{y}=0$
$\mathfrak{h}_2\oplus\mathfrak{h}_2$	$\dot{y}=0,\ \ddot{y}=0$
$\mathfrak{gl}(2)$	$\dot{y}=0$
$\mathfrak{sl}(2)\otimes\mathfrak{h}_2$	$\dot{y}=0,\ 2\dot{y}y^{(3)}-3\ddot{y}^2=0$
$\mathfrak{sl}(2)\otimes\mathfrak{sl}(2)\simeq\mathfrak{so}(2,2)$	$\dot{y}=0,\ 2\dot{y}y^{(3)}-3\ddot{y}^2=0$
$\mathfrak{sl}(2)$	$\dot{y}=0$

Higher dimensional spaces: time-dependent and independent symmetries

- transformation of the euclidean plane: no first order ODE;
- transformation of the hyperbolic plane:

$$\dot{x}=0,\quad \dot{y}=\frac{1}{xy-z},\quad \dot{z}=\frac{x}{xy-z};$$

- $so(3)$

$$\frac{dy}{dx}=\frac{y}{x},\qquad \frac{dz}{dx}=\frac{z}{x},$$

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References

[1] Olver, P.J.: *Applications of Lie Groups to Differential Equations*, Springer, 2000.

[2] Gonzalez-Lopez, A., Karman, N., Olver, P.J.: *Classification of Lie algebras of vector fields in the real plane*, Springer, 2000.

Thank you for your attention.

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