Invariant metrizability and projective metrizability on Lie groups

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Canonical connections on Lie groups

- G Lie group
- Canonical connections: ∇^-, ∇^+, ∇^0
- If X, Y are left-invariant vector fields:

$$\nabla^-_X Y = 0, \qquad \nabla^+_X Y = [X,Y], \qquad \nabla^0_X Y = \frac{1}{2}[X,Y],$$

- Curvature, torsion:

1. $R^- = 0$, $T^- \neq 0$,
2. $R^+ = 0$, $T^+ \neq 0$,
3. $T^0 = 0$, $R^0(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$,

- Geodesics: the 1-dimensional subgroups of the Lie group and their left (or right) translated images ($\Rightarrow$ canonical SODE of G).
- $\nabla^0 \;\Rightarrow\; S \;\Rightarrow\; \ddot{x}^i = f^i(x,\dot{x}) \;\Rightarrow\; \boxed{\ddot{x} = \dot{x}x^{-1}\dot{x}}$

Metrizability and projectively metrizability

Definition: A spray S (or a SODE) is

1) Riemann (resp. Finsler) metrizable, if there exists a Riemann (resp. Finsler) metric whose geodesics are the solution of S.
2) projectively Riemann (resp. Finsler) metrizable, if there exists a Riemann (resp. Finsler) metric whose geodesics are projectively equivalent to the solution of S.

Remark: the Euler-Lagrange equation inherits the symmetries of the Lagrangian:

$$\boxed{\text{left-invariant Lagrangian}} \quad \Rightarrow \quad \boxed{\text{left-invariant spray}}$$

The aim: find the relationship between invariant (Riemann or Finsler) metrizability, and the invariant (Riemann or Finsler) projectively metrizability of the canonical sprays for the class of Lie groups.

Metrizability, projectively metrizability and PDE

Euler-Lagrange equations:

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^i} - \frac{\partial \mathcal{L}}{\partial x^i} = 0} \Leftrightarrow \boxed{\omega = \left(\dot{x}^i \frac{\partial^2 \mathcal{L}}{\partial x^i \partial y^j} + \ddot{x} \frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j} - \frac{\partial \mathcal{L}}{\partial x^j} \right) dx^j = 0}$$

ℓ-homogeneity:

$$\boxed{F(x, \lambda y) = \lambda^\ell F(x, y)} \Leftrightarrow \boxed{y^i \frac{\partial F}{\partial y^i} = \ell \cdot F}$$

Proposition: The spray S is

1) *Riemann (resp. Finsler) metrizable* iff there exists a quadratic (resp. 2-homogeneous) regular function E on TM, such that the Euler-Lagrange PDE is satisfied.

2) *projectively Riemann (resp. Finsler) metrizable* iff there exists a quadratic (resp. 2-homogeneous) regular function $\overline{E}$ on TM, such that the Euler-Lagrange PDE is satisfied with $\overline{F} := \sqrt{2\overline{E}}$.

The geometric structure on a Lie group G

- Invariant coordinates:
  - $L_x : G \to G, \;\Rightarrow\; L_{x^{-1}} : T_x G \to \mathfrak{g} \;\Rightarrow\; T_x G \simeq \mathfrak{g} \;\Rightarrow\; TG \simeq G \times \mathfrak{g}$
  - Coordinates on TG: $(x, y) \;\Rightarrow\; (x, \alpha) \qquad (\alpha := L_{x^{-1}} dx)$

    Left translation: $L_g(x, y) = (gx, gy) \;\Leftrightarrow\; L_g(x, \alpha) = (gx, \alpha)$
  - Coordinates on TTG: $(x, \alpha) \to (x, \alpha, X, A)$:

$$(x, \alpha, X, A) \simeq X \frac{\partial}{\partial x}\Big|_{(x,\alpha)} + A \frac{\partial}{\partial \alpha}\Big|_{(x,\alpha)}$$

  - Left translation on TTG: $L_g(x, \alpha, X, A) = (gx, \alpha, gX, A)$

Using the "semi-invariant" coordinate system (x, α, X, A):

- $v(x, \alpha, xa, c) = (x, \alpha, 0, \frac{1}{2}[a, \alpha] + c),$
- $h(x, \alpha, xa, c) = (x, \alpha, xa, -\frac{1}{2}[a, \alpha])$
- $\tilde{a}^h|_{(x,\alpha)} = (x, \alpha, xa, -\frac{1}{2}[a, \alpha]) = xa\dfrac{\partial}{\partial x} - \frac{1}{2}[a, \alpha]\dfrac{\partial}{\partial \alpha}$
- $\tilde{a}^v|_{(x,\alpha)} = (x, \alpha, 0, a) = a\dfrac{\partial}{\partial \alpha}$
- $S(x, \alpha) = (x, \alpha, x\alpha, 0) = x\alpha\dfrac{\partial}{\partial x}$
- $\omega(\tilde{a}^h) = [a, \alpha]\dfrac{\partial E}{\partial \alpha} + x\alpha a\dfrac{\partial^2 E}{\partial x \partial \alpha} - xa\dfrac{\partial E}{\partial x}$

Invariant l-homogeneous Euler-Lagrange equation on Lie groups

Problem: Find a left invariant Lagrangian $F : TG \to \mathbb{R}$ which is ℓ-homogeneous and satisfies the Euler-Lagrange equation associated to the canonical spray of G.

PDE system I.

$$\frac{\partial F}{\partial x} = 0, \quad \alpha\frac{\partial F}{\partial \alpha} - \ell\cdot F = 0, \quad [a,\alpha]\frac{\partial F}{\partial \alpha} + x\alpha a\frac{\partial^2 F}{\partial x \partial \alpha} - xa\frac{\partial F}{\partial x} = 0$$

$\Leftrightarrow$

$$\frac{\partial F}{\partial x} = 0, \qquad \alpha\frac{\partial F}{\partial \alpha} - \ell\cdot F = 0, \qquad [a,\alpha]\frac{\partial F}{\partial \alpha} = 0, \quad \forall a \in \mathfrak{g},$$

PDE system II. (reduced system on $\mathfrak{g}$)

$$\alpha\frac{\partial \mathcal{F}}{\partial \alpha} - \ell\cdot \mathcal{F} = 0, \qquad [a,\alpha]\frac{\partial \mathcal{F}}{\partial \alpha} = 0, \quad \forall a \in \mathfrak{g},$$

Proposition: The canonical spray S of a Lie group is left invariant *projectively* Riemann (resp. Finsler) metrizable if and only if it is left invariant Riemann (resp. Finsler) metrizable.

$$[a,\alpha]\frac{\partial \mathcal{F}^2}{\partial \alpha} = 2\mathcal{F}\,[a,\alpha]\frac{\partial \mathcal{F}}{\partial \alpha} \quad \Rightarrow \quad \boxed{[a,\alpha]\frac{\partial (\mathcal{F}^2)}{\partial \alpha} = 0 \quad \Leftrightarrow \quad [a,\alpha]\frac{\partial \mathcal{F}}{\partial \alpha} = 0.}$$

Theorem: The canonical spray of a Lie group is left invariant projectively Finsler metrizable iff it is left invariant Riemann metrizable.

1) $\boxed{\text{Riemann}} \quad \Rightarrow \quad \boxed{\text{Finsler}} \quad \Rightarrow \quad \boxed{\text{projectively Finsler}}$

2) $\boxed{\text{projectively Finsler}} \quad \overset{prop}{\Rightarrow} \quad \boxed{\text{Finsler}} \quad \overset{Szabo}{\Rightarrow} \quad \boxed{\text{Riemann}}$

Remark:

$$\boxed{\text{left-invariant spray}} \quad \not\Rightarrow \quad \boxed{\text{left-invariant Lagrangian}}$$

Example: Heisenberg group $\mathbb{H}_3$

$$\mathbb{H}_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}, \qquad \mathfrak{h}_3 = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$[a,\alpha]\frac{\partial \mathcal{F}}{\partial \alpha}=0 \;\Rightarrow\; \frac{\partial \mathcal{F}}{\partial z}=0 \quad \Rightarrow \quad \text{no regular left-invariant Lagrangian}$$

But $\mathbb{H}_3$ is metrizable: $g = dx^2+dy^2+(dz-dx-dy)^2$

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