Non-existence of Funk functions for Finsler spaces of non-vanishing scalar flag curvature.

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Abstract

In [4, page 177], Zhongmin Shen asks "wether or not there always exist non-trivial Funk functions on a spray space". We will prove here that the answer is negative for the geodesic spray of a Finsler function of non-vanishing scalar flag curvature.

Résumé

Non-existence de fonction de Funk pour les espaces de Finsler de courbure scalaire non nulle. En [4, page 177], Zhongmin Shen demande s'il existe toujours une fonction non triviale de Funk sur un espace de spray. Ici, nous prouvons que la réponse est négative pour le spray géodésique d'une fonction de Finsler d'une courbure scalaire non nulle.

1. Introduction

A Funk function is a projective deformation of a homogeneous system of second order ordinary differential equations (SODE) that preserves the curvature (Jacobi endomorphism) of the given SODE.

If we start with a flat SODE, there are many corresponding Funk functions and each of them will projectively deform the given SODE into a Finsler metrizable one, [3, Theorem 7.1], [4, Theorem 10.3.5], solution to Hilbert's fourth problem.

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The existence of non-trivial Funk functions, for the general case of a non-flat SODE, is an open problem, as mentioned by Zhongim Shen in [4, page 177]. We will prove in Theorem 3.1 that there are non-trivial Funk functions for the geodesic equations of a Finsler function of non-vanishing scalar flag curvature.

2. Funk functions and projective deformations of Finsler spaces

We consider M a smooth n-dimensional manifold, TM its tangent bundle and $T_0M = TM \setminus \{0\}$ the tangent bundle with the zero section removed. Local coordinates on M are denoted by (x^i), while induced local coordinates on TM and T_0M are denoted by (x^i, y^i), for $i \in \{1, ..., n\}$.

Since most of the geometric objects that we will use in this paper are homogeneous with respect to the fibre coordinates on TM, we will assume them to be defined on T_0M. The homogeneity is characterised with the help of the Liouville (or dilation) vector field $\mathbb{C} = y^i\partial/\partial y^i$.

The geometric framework that we will use in this paper is based on the Frölicher-Nijenhuis formalism, [3, Chapter 2], [5]. Within this formalism one can associate to each vector valued form L, of degree ℓ, two derivations: i_L, of degree $(\ell - 1)$ and d_L, of degrees ℓ. For two vector valued forms K and L, of degree k and ℓ, we consider the Frölicher-Nijenhuis bracket $[K, L]$, which is a vector valued $(k + \ell)$-form, uniquely determined by $d_{[K,L]} = d_K d_L - (-1)^{k\ell} d_L d_K$. For various commutation formulae regarding the Frölicher-Nijenhuis theory, we refer to [3, Appendix A]. On TM there is a canonical vector valued 1-form, the tangent endomorphism $J = dx^i \otimes \partial/\partial y^i$ that induces two derivations i_J and d_J.

A system of second order ordinary differential equations on M,

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0, \tag{1}$$

can be identified with a vector field $S \in \mathfrak{X}(TM)$, $S = y^i\partial/\partial x^i - 2G^i(x, y)\partial/\partial y^i$, which satisfies $JS = \mathbb{C}$. Such a vector field is called a semispray. If additionally, $S \in \mathfrak{X}(T_0M)$ and satisfies the homogeneity condition $[\mathbb{C}, S] = S$ we say that S is a *spray*.

For a spray S, we can associate a nonlinear (Ehresman) connection with curvature tensors, [3, §3.1, §3.2]. We denote by h the horizontal projector, R the curvature of the connection and Φ the Jacobi endomorphism, given by

$$h = \frac{1}{2}\left(Id - [S, J]\right), \quad R = \frac{1}{2}[h, h], \quad \Phi = (Id - h) \circ [S, h].$$

The two curvature tensors R and Φ and the induced derivations are related by:

$$\Phi = i_S R, \quad [J, \Phi] = 3R, \quad 3d_R = d_{[J,\Phi]} = d_J d_\Phi + d_\Phi d_J.$$

A spray $S \in \mathfrak{X}(T_0M)$ is said to be *isotropic* if the induced Jacobi endomorphism takes the particular form

$$\Phi = \rho J - \alpha \otimes \mathbb{C}. \tag{2}$$

The function $\rho \in C^\infty(T_0M)$ is called the *Ricci scalar* and $\alpha = \alpha_i dx^i$ is a semi-basic 1-form (with respect to the canonical projection of the tangent bundle).

Definition 2.1 *A* Finsler function *is a continuous non-negative function $F : TM \to \mathbb{R}$ that satisfies the following conditions:*

i) F is smooth on T_0M and $F(x, y) = 0$ if and only if $y = 0$;

ii) F is positively homogeneous of order 1 in the fiber coordinates;

iii) the 2-form $dd_J F^2$ is a symplectic form on T_0M.

A spray $S \in \mathfrak{X}(T_0M)$ is said to be *Finsler metrizable* if there exists a Finsler function F that satisfies

$$i_S dd_J F^2 = -dF^2. \tag{3}$$

For a given Finsler function F, equation (3) uniquely determine a spray S, called the *geodesic spray* of the Finsler function F. The geometric structures associated to a Finsler function are those corresponding to its geodesic spray. Equation (3) is equivalent to $d_h F^2 = 0$.

A Finsler function F has *scalar flag curvature* $\kappa \in C^\infty(T_0M)$ if the Jacobi endomorphism is given by

$$\Phi = \kappa \left(F^2 J - F d_J F \otimes \mathbb{C}\right). \tag{4}$$

An orientation-preserving reparameterization of the system (1) gives rise to a new system and therefore a new spray $\widetilde{S} = S - 2P\mathbb{C}$, [4, Chapter 12]. These two sprays S and $\widetilde{S}$ are called *projectively related*. The function P is 1-homogeneous and it is called the projective deformation of the spray S.

Under a projective deformation $\widetilde{S} = S - 2P\mathbb{C}$, the corresponding geometric setting changes, the Jacobi endomorphisms being related by, [2, (4.8)],

$$\widetilde{\Phi} = \Phi + \left(P^2 - S(P)\right) J - \left(d_J(S(P) - P^2) + 3(P d_J P - d_h P)\right) \otimes \mathbb{C}. \tag{5}$$

From the above formula (5) we obtain that a projective deformation P preserves the Jacobi endomorphism if and only if it is a *Funk function*, which means that it satisfies

$$d_h P = P d_J P. \tag{6}$$

3. Non-existence results for Funk functions

We will prove now that there is an obstruction for the Funk equation (6), which is not satisfied for Finsler functions of non-vanishing scalar flag curvature.

Theorem 3.1 *Consider F a Finsler function F of non-vanishing scalar flag curvature κ. Then, there are no non-trivial Funk functions for the Finsler space (M, F).*

Proof. Consider S the geodesic spray of a Finsler function F, of non-vanishing scalar flag curvature κ. It follows that its Jacobi endomorphism is given by formula (4).

We will prove the statement of the theorem by contradiction. Therefore, we assume, that there exists a Funk function P, for the Finsler space (M, F). Hence, the function P satisfies the Funk equation (6).

If we apply the derivation d_J to both sides of (6) we obtain $d_J d_h P = 0$. Since $[J, h] = 0$, we obtain $0 = d_{[J,h]}P = d_J d_h P + d_h d_J P$ and hence $d_h d_J P = 0$. Using this formula and if we apply d_h to both sides of (6) we obtain

$$d_R P = d_h^2 P = d_h \left(P d_J P\right) = d_h P \wedge d_J P + P d_h d_J P = 0. \tag{7}$$

Due to the homogeneity of the involved geometric structures and the relations between the Jacobi endomorphism Φ and the curvature R of the nonlinear connection h, we obtain that the equation (7) is equivalent to

$$0 = i_S d_R P = d_{i_S R} P = d_\Phi P = 0. \tag{8}$$

Using the form (4) of the Jacobi endomorphism Φ, the above equation (8) can be written as follows

$$\kappa F^2 d_J P - \kappa F P d_J F = 0 \Longleftrightarrow d_J \left(\frac{P}{F}\right) = 0. \tag{9}$$

Last equation above implies that the function P/F is constant along the fibres the tangent bundle. Therefore, there exists a basic function a, on the base manifold M, such that

$$P(x,y) = a(x)F(x,y), \forall (x,y) \in T_0M. \tag{10}$$

Now, if we use the above form of the projective factor P and the fact that $d_hF = 0$, the Funk equation (6) can be written as follows

$$Fda = a^2 F d_J F \Longleftrightarrow d\left(\frac{-1}{a}\right) = d_J F. \tag{11}$$

The last equation (11) cannot have solutions since the left hand side is a basic 1-form, while the right hand side is a semi-basic (and not basic) 1-form. □

We can reinterpret Theorem 3.1 as follows. Consider S the geodesic spray of a Finsler function of non-vanishing scalar flag curvature. Then there is no projective deformation of the spray S, by a projective factor P, which preserves the curvature of the spray.

In [1, Theorem 3.1] it is shown that for a Finsler function of non-vanishing scalar flag curvature, the projective class of its geodesic spray contains exactly one spray with the given curvature tensor. Next corollary, which is a consequence of Theorem 3.1, gives a new motivation for [1, Theorem 3.1].

Corollary 3.2 *Consider S an isotropic spray of non-vanishing Ricci scalar. Then there is no Funk function on the spray space (M, S) that will projectively deform S into a Finsler metrizable spray.*

Proof. Consider S an isotropic spray of non-vanishing Ricci scalar. By contradiction, we assume that there is a Funk function P, for the spray space (M, S), such that the projectively deformed spray $\widetilde{S} = S - 2P\mathbb{C}$ is Finsler metrizable by a Finsler function F. Since P is a Funk function it follows that the two sprays S and $\widetilde{S}$ have the same Jacobi endomorphisms and hence $\widetilde{S}$ is isotropic as well, with non-vanishing Ricci scalar. Since $\widetilde{S}$ is Finsler metrizable by a Finsler function F, it follows that F is of non-vanishing scalar flag curvature. It follows that $-P$ is a Funk function for the Finsler space (M, F), which contradicts the result of Theorem 3.1. □

The results of Theorem 3.1 and Corollary 3.2 heavily rely on the fact that the curvature of the considered spray and Finsler function does not vanish. This implies that for the existence of the Funk function, we have the obstructions (7) and (8). In the flat case, these obstructions are automatically satisfied, and it explains why in this case there are many Funk functions and each of them leads to sprays that are Finsler metrizable, [3, Theorem 7.1], [4, Theorem 10.3.5].

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References

[1] Bucataru, I.: Funk functions and projective deformations of sprays and Finsler spaces of scalar flag curvature, arXiv:1412.5282.

[2] Bucataru, I., Muzsnay, Z.: *Projective and Finsler metrizability: parametrization rigidity of geodesics*, Int. J. Math., **23** no. 6 (2012), 1250099.

[3] Grifone, J., Muzsnay, Z.: *Variational Principles for Second-order Differential Equations*, World-Scientific, 2000.

[4] Shen, Z.: Differential geometry of spray and Finsler spaces, Springer, 2001.

[5] Szilasi, J.: *A setting for spray and Finsler geometry*, in "Handbook of Finsler Geometry" (ed. P.L. Antonelli), Kluwer Acad. Publ., Dordrecht, **2** (2003), 1183–1426.