About Finsler holonomy, metrizability

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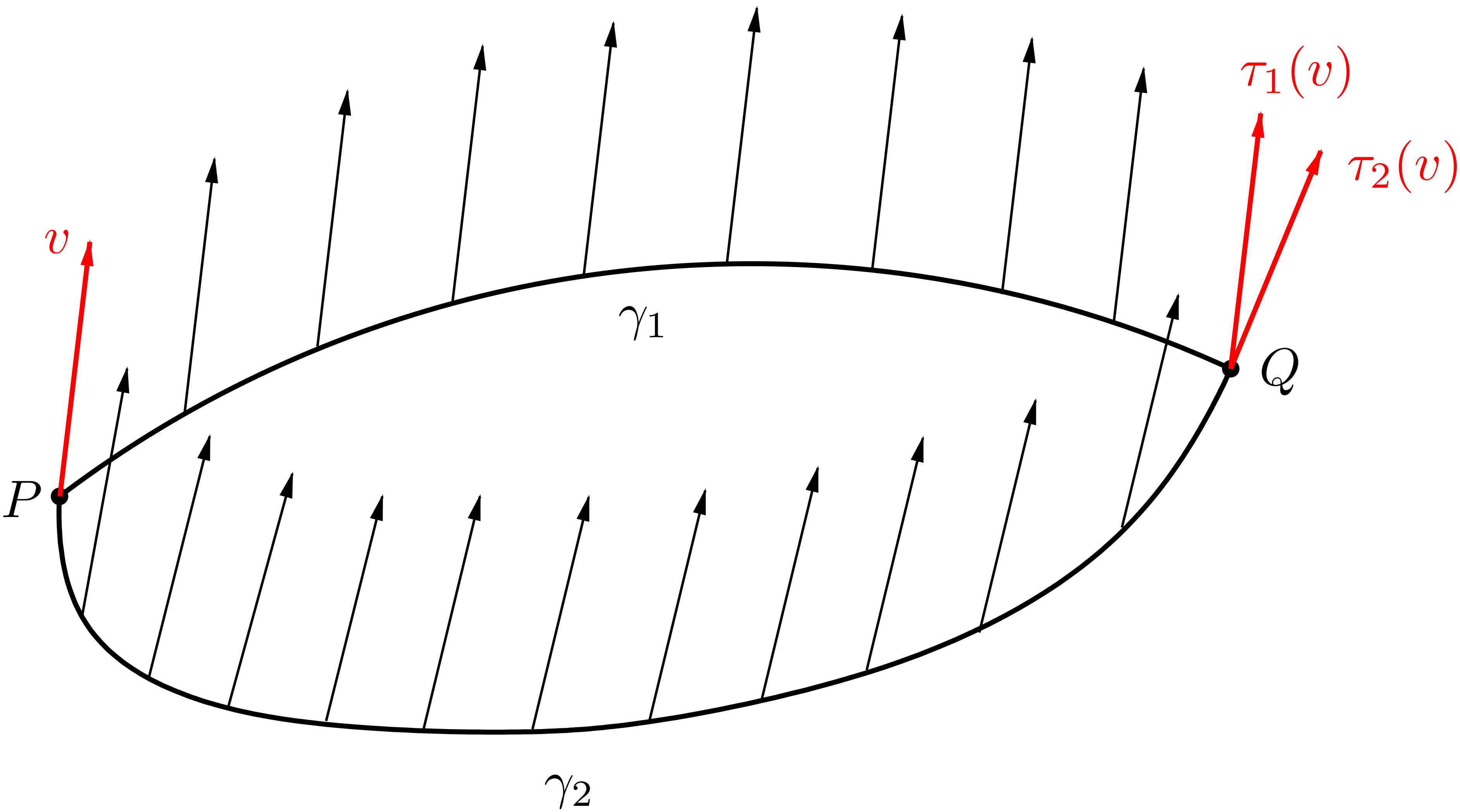
February 11, 2015, Debrecen

Abstract

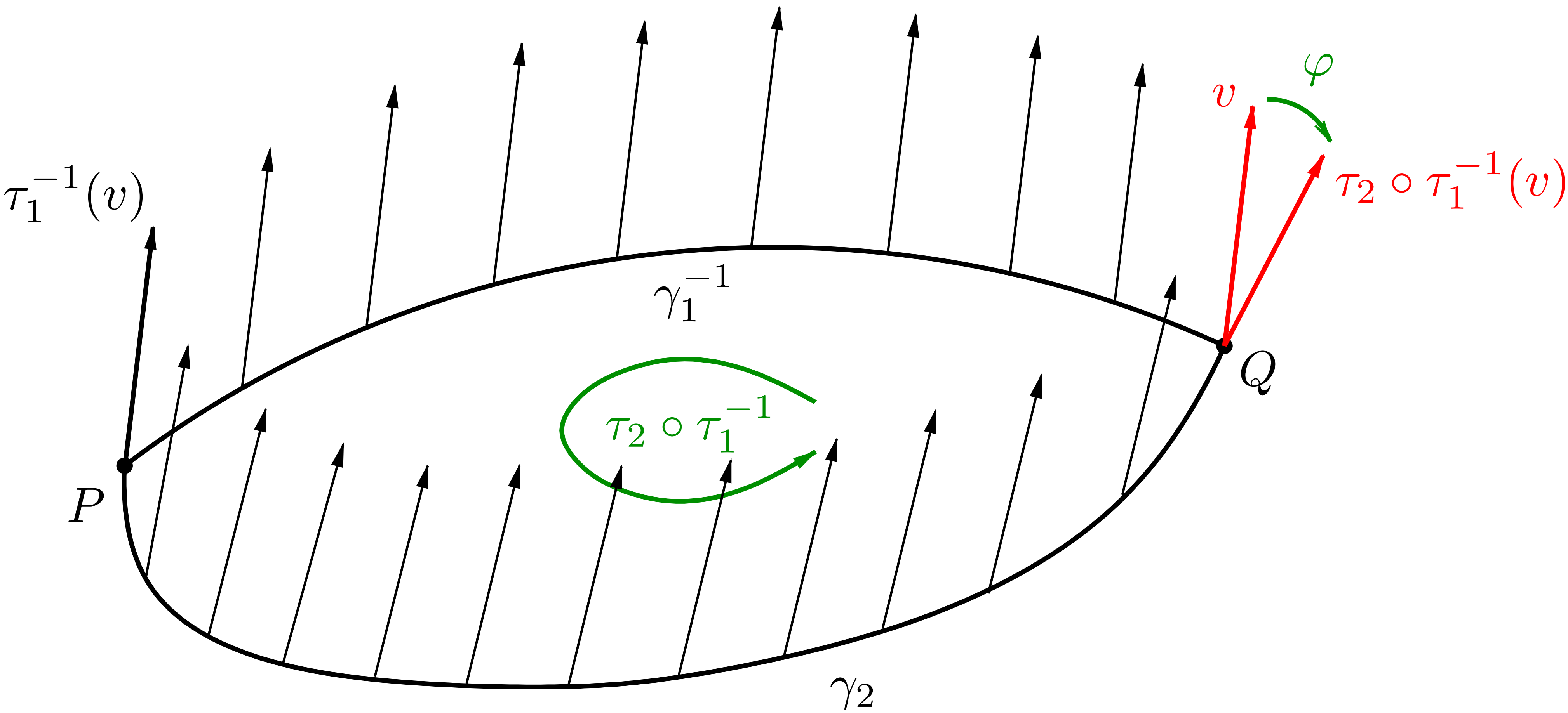
The aim of this talk is to present some results about

- the holonomy of Finsler manifolds (P.T. Nagy),
- metrizability by Finsler functions of constant/scalar flag curvature (I. Bucataru),
- invariant metrizability and projective metrizability on Lie groups (T. Milkovszki)

Parallel translation along a curves



Holonomy



$\tau_1^{-1}(v)$
v
φ
$\tau_2 \circ \tau_1^{-1}(v)$
γ_1^{-1}
$\tau_2 \circ \tau_1^{-1}$
P
Q
γ_2

Parallel translation, holonomy

- Finslerian metric: $g_{ij}(\mathbf{x},\mathbf{y})dx^i \otimes dx^j$

- Geodesics: $\ddot{x}^i + 2G^i(x,\dot{x}) = 0, \qquad G^i := \frac{1}{4}g^{il}\Big(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\Big)y^j y^k.$

- Parallel vector field $X(t)$ along a curve $c(t)$:

$$\nabla_{\dot{c}}X(t) = \Big(\frac{dX^i(t)}{dt} + \Gamma^i_j(c(t),X(t))\dot{c}^j(t)\Big)\frac{\partial}{\partial x^i} = 0, \quad \Gamma^i_j = \frac{\partial G^i}{\partial y^j}.$$

- Parallel translation along a curve $c\colon[0,1]\to M$:

$$\tau_c\colon T_{c_0}M \to T_{c_1}M, \qquad \Rightarrow \qquad \tau_c : \mathcal{I}_{c_0} \to \mathcal{I}_{c_1}$$

- The *holonomy group* is generated by parallel translation along closed curves

$\Rightarrow$ subgroup of $\mathrm{Diff}^\infty(\mathcal{I}_x)$ determined by parallel translations.

Tangent Lie algebras to a subgroup H of $\mathrm{Diff}^{\infty}(\mathcal{I})$

Def:
- A vector field X is ***tangent*** to H, if there exists a differentiable curve of diffeomorphisms $\{\phi_t\}$ in H such that

$$\phi_0 = \mathrm{Id}, \qquad \left.\frac{\partial \phi_t}{\partial t}\right|_{t=0} = X.$$

- A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{X}^{\infty}(\mathcal{I})$ is called ***tangent*** to H, if all elements of $\mathfrak{h}$ are tangent to H.

$$\boxed{\mathfrak{h} \text{ tangent to } H \quad \Rightarrow \quad \text{information on } H}$$

Property: If $\mathfrak{h}$ is tangent to a closed subgroup H, then

$$\boxed{\exp(\mathfrak{h}) \subset H}$$

Curvature algebra and infinitesimal holonomy algebra

$\mathfrak{R}(M)$ - *curvature algebra*: is the smallest Lie algebra generated by curvature vector fields.

$\mathfrak{hol}^*(M)$ - *infinitesimal holonomy algebra*: is the smallest Lie algebra generated by curvature vector fields and by horizontal Berwald differentiation.

If $x \in M$,

$$\mathfrak{R}_x(M) = \big\{\, \xi(x)\,;\, \xi \in \mathfrak{R}(M) \,\big\} \qquad \subset \mathfrak{X}^\infty(\mathcal{I}_x)$$

$$\mathfrak{hol}^*_x(M) = \big\{\, \xi(x)\,;\, \xi \in \mathfrak{hol}^*(M) \,\big\} \qquad \subset \mathfrak{X}^\infty(\mathcal{I}_x)$$

Proposition: $\mathfrak{R}_x(M)$ and $\mathfrak{hol}^*_x(M)$ are tangent to $\mathrm{Hol}_x(M)$.

Holonomy of Finsler surfaces ($\dim M = 2$)

- $\mathcal{I}_x \simeq \mathbb{S}^1$, $\quad \mathsf{Hol}_x(M) \subset \mathsf{Diff}^\infty_+(\mathbb{S}^1)$
- $\dim \mathfrak{R}_x(M) \leq 1$,
- $\mathfrak{hol}^*_x(M)$ can be higher (even infinite) dimensional.

Projectively flat Finsler manifolds of constant curvature:

$$G^i = \mathcal{P}(x,y)y^i, \qquad R^i_{jk} = \lambda\big(\delta^i_j g_{km} y^m - \delta^i_k g_{jm} y^m\big),$$

Theorem. (Finsler 2-manifolds with maximal holonomy group)
If there exists $x_0 \in M$ with $\mathcal{F}(x_0,y) = \|y\|$ and $\mathcal{P}(x_0,y) = c\cdot\|y\|$, then

$$\overline{\mathsf{Hol}_{x_0}(M)} = \mathsf{Diff}^\infty_+(\mathbb{S}^1).$$

Idea:

$$\mathcal{I}_0 = \mathbb{S}^1, \qquad \Big\{\cos nt\,\frac{\partial}{\partial t}, \sin nt\,\frac{\partial}{\partial t}\Big\}_{n\in\mathbb{N}} \subset \mathfrak{hol}^*_0(M) \quad \Rightarrow \quad \overline{\mathfrak{hol}^*_0(M)} = \mathfrak{X}(\mathbb{S}^1)$$

$$\mathsf{Diff}^\infty_+(\mathbb{S}^1) \subset \overline{\mathsf{Hol}_0(M)} \subset \mathsf{Diff}^\infty_+(\mathbb{S}^1)$$

Characterization of the holonomy of projectively flat Finsler manifolds of constant curvature

Theorem: The holonomy group of a locally projectively flat Finsler manifold of constant curvature λ is finite dimensional if and only if

1. $\lambda = 0$,
2. $\lambda \neq 0$ and the connection is linear (the Finsler space is a Berwald space).

If $\lambda = 0, \quad \Rightarrow \quad R = 0, \quad \Rightarrow \quad \mathsf{Hol}(M) = \{\mathrm{id}\}$

If $\lambda \neq 0$,

- ∇ linear $\quad \Rightarrow \quad \mathsf{Hol}(M) \subset GL(n, \mathbb{R}) \quad \Rightarrow \quad \dim \mathsf{Hol}(M) < \infty$.
- ∇ is nonlinear, $\quad \Rightarrow \quad \mathsf{Hol}(M)$ is infinite dimensional.

Inverse problem of the calculus of variations, metrizability and projective metrizability

Inverse problem: under what conditions the solutions of a system of SODE

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \dot{x}) = 0, \ i \in \{1, 2, ..., n\}, \tag{1}$$

are among the solutions of the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0, \ i \in \{1, 2, ..., n\}, \tag{2}$$

for some Lagrangian function L.

Homogeneous SODE (spray)

- $L = F^2$ (F Finsler function) $\Longrightarrow$ Finsler metrizability problem,
- $L = F \Longrightarrow$ projective metrizability problem.

Main results

- In [1] we provide the necessary and sufficient conditions one has to check in order to decide if a given spray is metrizable by a Finsler function of *constant flag curvature*.

- In [2] we provide the necessary and sufficient conditions one has to check in order to decide if a given spray is metrizable by a Finsler function of *scalar flag curvature*.

- Both theorems provide an algorithm to construct the Finsler function that metricizes a given spray, in the case that is variational.

The geometric setting induced by a spray

Jacobi endomorphism: $\Phi = v \circ \mathcal{L}_S h = \mathcal{L}_S h \circ h$.

$$\Phi = R^i_j \frac{\partial}{\partial y^i} \otimes dx^j = \left(2\frac{\partial G^i}{\partial x^j} - S\left(\frac{\partial G^i}{\partial y^j}\right) - \frac{\partial G^i}{\partial y^r}\frac{\partial G^r}{\partial y^j} \right) \frac{\partial}{\partial y^i} \otimes dx^j.$$

Ricci scalar, ρ,

$$\mathrm{Tr}(\Phi) = R^i_i = (n-1)\rho$$

A spray S is *isotropic* if its Jacobi endomorphism has the form,

$$\Phi = \rho J - \alpha \otimes \mathbb{C},$$

F is said to be of *constant/scalar flag curvature* if there exists a constant/scalar function κ on $T_0 M$, such that

$$\Phi = \kappa \left(F^2 J - F d_J F \otimes \mathbb{C} \right).$$

Sprays metrizable by Finsler functions of constant curvature

Theorem: Consider S a spray with non-vanishing Ricci scalar. The spray S is metrizable by a Finsler function of non-zero constant flag curvature if and only if its Jacobi endomorphism satisfies:

$$
\begin{aligned}
AC) &\quad \operatorname{rank} dd_J(\operatorname{Tr}\Phi) = 2n \\
C_1) &\quad 2(n-1)\Phi - 2(\operatorname{Tr}\Phi)J + d_J(\operatorname{Tr}\Phi)\otimes\mathbb{C} = 0; \\
C_2) &\quad d_h(\operatorname{Tr}\Phi) = 0.
\end{aligned}
$$

Remarks:

- $F^2 = \frac{1}{n-1}\operatorname{sign}(\operatorname{Tr}\Phi)\operatorname{Tr}\Phi$.
- $AC \Leftrightarrow$ regularity condition of the Finsler function
- $C_1 \Leftrightarrow S$ is isotropic and $d_J\alpha = 0$.
- $C_2 \Leftrightarrow S$ is Ricci constant.

dim $M \geq 3$

Theorem: Consider S a spray of non-vanishing Ricci scalar, satisfying the condition AC and C_1 (algebraic condition + isotropy + $d_J\alpha = 0$). Then the following five conditions are equivalent:

i) S is Finsler metrizable;

ii) S is metrizable by a Finsler metric of non-vanishing scalar flag curvature;

iii) S is Finsler metrizable by an Einstein metric;

iv) S is metrizable by a Finsler metric of non-zero constant flag curvature;

v) S is Ricci constant.

Sprays metrizable by Finsler functions of scalar curvature

Theorem: Consider S a spray of non-vanishing Ricci scalar. Then S is metrizable by a Finsler function F of scalar flag curvature if and only if

$S_1)$ S is isotropic;
$S_2)$ $d_J(\alpha/\rho) = 0$;
$S_3)$ $D_{hX}(\alpha/\rho) = 0$, for all $X \in \mathfrak{X}(T_0M)$;
$AS)$ $d(\alpha/\rho) + 2i_{\mathbb{F}}\alpha/\rho \wedge \alpha/\rho$ is a symplectic form on T_0M.

Idea: $S_2 \implies \exists f$, on T_0M, such that

$$\frac{1}{\rho}\alpha = d_J f$$

$S_3 \implies \exists\ a$, on M such that

$$d_h f = da = d_h a \qquad \Longrightarrow \qquad F = \exp(f - a)$$

Canonical connections on Lie groups

- G Lie group
- Canonical connections: ∇
- If X, Y are left-invariant vector fields: $\nabla_X Y = \frac{1}{2}[X, Y]$,
- Torsion, curvature, : $T = 0$, $R(X, Y)Z = -\frac{1}{4}[[X, Y], Z]$,
- Geodesics: the 1-dimensional subgroups of the Lie group and their left or right translated images

Remark: the Euler-Lagrange equation inherits the symmetries of the Lagrangian:

$$\boxed{\text{invariant Lagrangian}} \quad \Rightarrow \quad \boxed{\text{invariant spray}}$$

The aim: find the relationship between invariant Riemann/Finsler metrizability, and the invariant Riemann/Finsler projectively metrizability of the canonical sprays for the class of Lie groups.

Proposition: The canonical spray S of a Lie group is left invariant *projectively* Riemann (resp. Finsler) metrizable if and only if it is left invariant Riemann (resp. Finsler) metrizable.

$$\boxed{[a,\alpha]\frac{\partial(\mathcal{F}^2)}{\partial\alpha}=0 \quad \Leftrightarrow \quad [a,\alpha]\frac{\partial\mathcal{F}}{\partial\alpha}=0.}$$

Theorem: The canonical spray of a Lie group is left invariant projectively Finsler metrizable iff it is left invariant Riemann metrizable.

1) $\boxed{\text{Riemann}} \Rightarrow \boxed{\text{Finsler}} \Rightarrow \boxed{\text{projectively Finsler}}$

2) $\boxed{\text{projectively Finsler}} \overset{prop}{\Rightarrow} \boxed{\text{Finsler}} \overset{Szabo}{\Rightarrow} \boxed{\text{Riemann}}$

References

[1] Ioan Bucataru, Zoltán Muzsnay: *Sprays metrizable by Finsler functions of constant flag curvature*, Differential Geom. Appl., vol. 31, no. 3 (2013), 405-415,

[2] Ioan Bucataru, Zoltán Muzsnay: *Metrizable isotropic second-order differential equations and Hilbert's fourth problem*, Journal of the Aust Math. Soc., 97, 1, 2014, pp 27-47,

[3] Tamás Milkovszki, Zoltán Muzsnay: *Invariant metrizability and projective metrizability on Lie groups*,

[4] Zoltán Muzsnay, Péter T. Nagy: *Finsler 2-manifolds with maximal holonomy group of infinite dimension* Differential Geometry and its Applications, doi:10.1016/j.difgeo.2015.01.001

Acknowledgements: The research by the authors leading to these results was funded in part by the European Union's Seventh Framework Programme (FP7/2007-2013) under grant agreement no. 318202.