A Finsler terek holonómiájáról

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Új eredmények

- metrizálhatóság: konstans/skalár göbületű Finsler metrika (I. Bucataru),
- Finsler terek holonómiája (P.T. Nagy),

Parallel translation, holonomy

- Finslerian metric: $g = g_{ij}(\mathbf{x},\mathbf{y})dx^i \otimes dx^j$

- Geodesics: $\ddot{x}^i + 2G^i(x,\dot{x}) = 0,$ $\quad G^i := \frac{1}{4}g^{il}\Big(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\Big)y^j y^k.$

- Parallel vector field $X(t)$ along a curve $c(t)$:

$$\nabla_{\dot{c}}X(t) = \Big(\frac{dX^i(t)}{dt} + \Gamma^i_j(c(t),X(t))\dot{c}^j(t)\Big)\frac{\partial}{\partial x^i} = 0, \quad \Gamma^i_j = \frac{\partial G^i}{\partial y^j}.$$

- Parallel translation along a curve $c\colon[0,1]\to M$:

$$\tau_c\colon T_{c_0}M \to T_{c_1}M, \quad \Rightarrow \begin{cases} \tau(\lambda v) = \lambda\tau(v) \\ \|\tau(v)\| = \|\tau(v)\| \end{cases} \Rightarrow \tau_c : \mathcal{I}_{c_0} \to \mathcal{I}_{c_1}$$

- The *holonomy group* is generated by parallel translation along closed curves

$\Rightarrow$ | subgroup of $\mathrm{Diff}^\infty(\mathcal{I}_x)$ determined by parallel translations. |

Tangent Lie algebras to a subgroup H of $\mathrm{Diff}^\infty(\mathcal{I})$

Def: • A vector field X is ***tangent*** to H, if there exists a differentiable curve of diffeomorphisms $\{\phi_t\}$ in H such that

$$\phi_0 = \mathrm{Id}, \qquad \left.\frac{\partial \phi_t}{\partial t}\right|_{t=0} = X.$$

• A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{X}^\infty(\mathcal{I})$ is called ***tangent*** to H, if all elements of $\mathfrak{h}$ are tangent to H.

$$\boxed{\mathfrak{h} \text{ tangent to } H \quad \Rightarrow \quad \text{information on } H}$$

Property: If $\mathfrak{h}$ is tangent to a closed subgroup H, then

$$\boxed{\exp(\mathfrak{h}) \subset H}$$

Definition: A vector field X is ***strongly tangent*** to a subgroup H, if there exists a $k \in \mathbb{N}$ and a smooth k-parameter family $\{\phi_{(t_1,\dots,t_k)}\}$ of diffeomorphisms in H such that

1. $\phi_{(t_1,\dots,t_k)} = \mathrm{Id}$, if $t_j = 0$ for some $1 \leq j \leq k$;
2. $\left.\dfrac{\partial^k \phi_{(t_1,\dots,t_k)}}{\partial t_1 \dots \partial t_k}\right|_{(t_1,\dots,t_k)=(0,\dots,0)} = X.$

Proposition: *The Lie algebra generated by strongly tangent vector fields is tangent to H.*

Proposition: $\mathfrak{R}_x(M)$ and $\mathfrak{hol}^*_x(M)$ are tangent to $\mathrm{Hol}_x(M)$.

- $\mathfrak{R}(M)$: the *curvature algebra* is the smallest Lie algebra generated by curvature vector fields.
- $\mathfrak{hol}^*(M)$: the *infinitesimal holonomy algebra* is the smallest Lie algebra generated by curvature vector fields and by horizontal Berwald differentiation.

Projectively flat Finsler manifolds of constant curvature:

$$G^i = \mathcal{P}(x,y)y^i, \qquad R^i_{jk} = \lambda\left(\delta^i_j g_{km} y^m - \delta^i_k g_{jm} y^m\right),$$

Theorem. The holonomy group of a projectively flat, spherically symmetric Finsler 2-manifolds of constant curvature is maximal:

$$\overline{\mathsf{Hol}_{x_0}(M)} = \mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1).$$

Theorem: The holonomy group of a locally projectively flat Finsler manifold of constant curvature is finite dimensional if and only if

1. $R = 0$,
2. $R \neq 0$ and the associated canonical connection is linear.

S. Lie: Ha egy véges dimenziós Lie csoport fixpont nélkül hat egy 1-dimenziós sokaságon, akkor annak dimenziója maximum 3.

Tegyük fel, hogy $R \neq 0$.

- Ha ∇ lineáris $\Rightarrow$ Hol(M) lineáris $\Rightarrow$ véges dimenziós.
- Ha ∇ nem lineáris,

  tegyük fel, hogy Hol(M) véges dimenziós, $x_0 \in M$, $\mathcal{R} = R_{x_0}(X,Y)$

  · $\mathcal{P}(x_0, y)$ nem lineáris az y-ban $\Rightarrow$ $\{\mathcal{R},\ \nabla_1\mathcal{R},\ \nabla_2\mathcal{R}\}$ lineárisan független,

  · $\{\mathcal{R},\ \nabla_1\mathcal{R},\ \nabla_2\mathcal{R},\ \nabla_i\nabla_j\mathcal{R}\}$ lineárisan függő,

  · $\left\{\Phi_0 = 1,\ \Phi_1 = \frac{\partial\mathcal{P}}{\partial y^1},\ \Phi_2 = \frac{\partial\mathcal{P}}{\partial y^2},\ \Phi_{ij} = 2\frac{\partial\mathcal{P}}{\partial y^i}\frac{\partial\mathcal{P}}{\partial y^j} - \lambda g_{ij}\right\}$ lineárisan függő

  · $g_{ij} = A_{ij} + B^1_{ij}\mathcal{P}_1 + B^2_{ij}\mathcal{P}_2 + \frac{2}{\lambda}\mathcal{P}_i\mathcal{P}_j,$ $\qquad A_{ij}, B^1_{ij}, B^2_{ij} \in \mathbb{R}$,

  · $g_{ij} = \partial_{y^i y^j}E \ \Rightarrow\ \partial_i g_{jk} - \partial_k g_{ij} = 0$ $\Rightarrow$ PDE $\mathcal{P}$-re

  · $$\begin{cases} 2\mathcal{P}_2\mathcal{P}_{11} - 2\mathcal{P}_1\mathcal{P}_{12} + b_2\mathcal{P}_{11} + (c_2 - b_1)\mathcal{P}_{12} - c_1\mathcal{P}_{22} = 0, \\ 2\mathcal{P}_1\mathcal{P}_{22} - 2\mathcal{P}_2\mathcal{P}_{12} - b_3\mathcal{P}_{11} + (b_2 - c_3)\mathcal{P}_{12} + c_2\mathcal{P}_{22} = 0. \end{cases}$$

$$\mathcal{P}(x_0,y) = y_2 \cdot f\left(\frac{y_1}{y_2}\right) \Rightarrow \begin{cases} f''\left(\frac{2}{y_2} f + \frac{b_2}{y_2} + (b_1 - c_2)\frac{y_1}{y_2^2} + \frac{c_1 y_1^2}{y_2^3}\right) = 0, \\ f''\left(\frac{2y_1}{y_2^2} f - \frac{b_3}{y_2} + (c_3 - b_2)\frac{y_1}{y_2^2} + \frac{c_2 y_1^2}{y_2^3}\right) = 0. \end{cases}$$

$$\begin{cases} 2f + b_2 + (b_1 - c_2)t + c_1 t^2 = 0, \\ 2tf - b_3 + (c_3 - b_2)t + c_2 t^2 = 0. \end{cases}$$

$$b_3 + (2b_2 - c_3)t - (2c_2 - b_1)t^2 + c_1 t^3 \equiv 0,$$

$$b_3 = 0, \qquad 2b_2 - c_3 = 0, \qquad 2c_2 - b_1 = 0, \qquad c_1 = 0.$$

$$f(t) = -b_2 - c_2 t, \qquad \mathcal{P}(x_0,y) = -y_2 b_2 - c_2 y_1$$

$\Rightarrow$ $\mathcal{P}$ lineáris $\Rightarrow$ ∇ lineáris $\Rightarrow$ ellentmondás.

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