Invariant metrizability and projective metrizability on Lie groups

Tamás Milkovszki, Zoltán Muzsnay

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- Metrizability: a homogeneous SODE $x^i = f^i(x, \dot{x})$ is Riemann (resp. Finsler) metrizable, if there exists a Riemann (resp. Finsler) metric whose geodesics are the solution of the SODE (spray).
- Projective metrizability: a homogeneous SODE is projectively Riemann (resp. Finsler) metrizable, if there exists a Riemann (resp. Finsler) metric such that the geodesics of the SODE and the geodesics of the manifold coincide up to an orientation preserving reparametrisation.

- Every Riemann metrizable (resp. proj. metrizable) SODE is also Finsler metrizable (resp. proj. metrizable).
- The class of the Finsler metrizable sprays are larger than the Riemann metrizable sprays.
  In the case of the quadratic sprays (linear connections) the two problem coincide due to Szabó's theorem.
- The class of the projective Finsler metrizable sprays (even in the quadratic (linear) case) are larger than the projective Riemann metrizable sprays.
- **Main theorem**: the canonical connection of a Lie group is left invariant projectively Finsler metrizable if and only if it is Riemann metrizable.

M will denote an n-dimensional smooth manifold, TM its tangent bundle.
The i_* and the d_* type derivation associated to a vector valued l-form L will be denoted by i_L and d_L. They can be defined in the following way: if $\deg L = 0$, i.e. $L \in \mathfrak{X}(M)$, then $i_L := \omega(L)$. If $degL = l > 1$, then

$$i_L \omega(X_1, \ldots, X_l) = \omega(L(X_1, \ldots, X_l)),$$

and:

$$d_L := [i_L, d] = i_L d - (-1)^{l-1} d i_L$$

- The *vertical endomorphism* (tangent strucure) is a vector-valued 1-form, locally is given by:

$$J = \frac{\partial}{\partial y^i} \otimes dx^i$$

- The *Liouville vector field* locally has the form:

$$C = y^i \frac{\partial}{\partial y^i}$$

- $F : TM \to \mathbb{R}$ function is a k-homogeneous function $(F(x, \lambda y) = \lambda^k F(x, y))$ if and only if

$$CF = y^i \frac{\partial}{\partial y^i} = kF.$$

- A vector field $S \in \mathfrak{X}(TM)$ is called a *spray* if $JS = C$ and $[C, S] = S$. The coordinate representation of the spray S:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

  where $G^i(x, y)$ functions are homogeneous of degree 2 in $y = (y^j)$, that is $G^i(x, \lambda y) = \lambda^2 Gi(x, y), \lambda > 0$.
  A spray is called *quadratic* if the spray coefficients are quadratic functions in y, that is $G^i = \frac{1}{2}\Gamma^i_{jk} y^j y^k$.
- The curve $\gamma : I \to M$ is a *geodesic of a spray* S if $S \circ \gamma' = \gamma''$.
- The geodesics of the spray S are the solutions of the SODE
  $$\ddot{x}^i + 2G^i(x, \dot{x}) = 0, \qquad i = 1, ..., n.$$

- A connection can be associated to every S. This connection is $\Gamma := [J, S] \in \Psi^1(TM)$. Then

$$VTM = \{X \in \mathfrak{X}(TM) | \Gamma(X) = -X\}$$
$$HTM = \{X \in \mathfrak{X}(TM) | \Gamma(X) = X\}$$

and $VTM \oplus HTM = TTM$

- The *horizontal* and *vertical endomorphism* associated to S:

$$h := \frac{1}{2}(Id_{\mathfrak{X}(TM)} + \Gamma) \qquad v := \frac{1}{2}(Id_{\mathfrak{X}(TM)} - \Gamma)$$

- (M, F) manifold is a *Finsler-manifold* if $F : TM \to \mathbb{R}_+$ is a continuous function which is smooth on $TM \setminus \{0\}$ such that
    1. F fiberwise positive homogeneous of degree one, that is $F(x, \lambda y) = \lambda F(x, y), \quad \lambda > 0.$
    2. $$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$$
       positive definite.
- The geodesics of the Finsler manifold are the extremals of the arc-lenght functional. They are the solutions of the Euler-Lagrange differential equation.

- If (M, F) is a Finsler-manifold, then there exits a unique spray S over $\mathcal{T}M := TM \setminus \{0\}$ (canonical spray), which is determined by the following formula:

$$i_S dd_J E = -dE,$$

where $E = \frac{1}{2}F^2$.
- S spray on M is Finsler (resp. Riemann) metrizable if there exist a Finsler (resp. Riemann) metric on M whose canonical spray is S.

- Two sprays S and S' are projectively equivalent if the geodesics of S and the geodesics of S' coincide up to an orientation preserving reparametrisation.
- S and S' are projectively equivalent, if there exist a function $\lambda \in C^\infty(\mathcal{T}M)$ such that $S' = S + \lambda C$.
- S spray on M is projectively Finsler (resp. Riemann) metrizable if there exist a Finsler (resp. Riemann) metric on M whose canonical spray is projectively equivalent to S.

- To every Lagrangian $\mathcal{L} : TM \to \mathbb{R}$ a 1-form

$$\omega_{\mathcal{L}} := i_S dd_J \mathcal{L} + dd_C \mathcal{L} - d\mathcal{L}$$

can be associated (Euler–Lagrange form).

- We have

$$\omega_{\mathcal{L}} = 0 \quad \Leftrightarrow \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^i} - \frac{d\mathcal{L}}{dx^i} = 0.$$

That is, $\omega_{\mathcal{L}} = 0$ is the coordinate free version of the Euler-Lagrange equation.

- S spray is Riemann (resp. Finsler) metrizable if and only if there exists a quadratic (resp. 2-homogeneous) function $E : TM \to \mathbb{R}$, such that $\left(\frac{\partial^2 E}{\partial y^i \partial y^j}\right)$ is positive definite on $TM \setminus \{0\}$ and the Euler-Lagrange PDE is satisfied.
- S spray is projectively Riemann (resp. Finsler) metrizable if and only if there exists a quadratic (resp. 2-homogeneous) function $\overline{E} : TM \to \mathbb{R}$, such that such that $\left(\frac{\partial^2 \overline{E}}{\partial y^i \partial y^j}\right)$ is positive definite on $TM \setminus \{0\}$ and the Euler-Lagrange PDE is satisfied with $\mathcal{L} := \sqrt{2\overline{E}}$.

- Let $M = G$ be a Lie group,
  $L_g : G \to G, L_g(\widetilde{g}) := g\widetilde{g}$ the left translation,
  $\alpha : T_g G :\to \mathfrak{g}, \alpha(v) := (L_{g^{-1}})_* v$ is the Maurer-Cartan form.
- We will use the (x^i, α^i), $\alpha^i = (L_{x^{-1}*})^i_j dx^j$,,left invariant" coordinate system on $TG \cong G \times \mathfrak{g}$.
- A vector field X is left invariant, if $(L_g)_* X = X, \forall g \in G$.
  A function $\mathcal{L} : TG \to \mathbb{R}$ is left invariant, if $\mathcal{L} \circ (L_g)_* = \mathcal{L}$.
- Remark: $\mathcal{L} : TG \to \mathbb{R}$ is left invariant $\Leftrightarrow \frac{\partial \mathcal{L}}{\partial x^i} = 0, i = 1, ..., n$.

Using the coordinate system (x, α) we obtain the following theorem:

Theorem

The Lagrangian $\mathcal{L} : TG \to \mathbb{R}$ is a left-invariant k-homogeneous solution to the Euler-Lagrange equation associated to the canonical spray of the Lie group G, if and only if

$$\frac{\partial \mathcal{L}}{\partial x^i} = 0, \quad i = 1, ..., n \tag{1}$$

$$\alpha^i \frac{\partial \mathcal{L}}{\partial \alpha^i} = k\mathcal{L}, \tag{2}$$

$$[a, \alpha]^i \frac{\partial \mathcal{L}}{\partial \alpha^i} = 0, \quad \forall a \in \mathfrak{g}. \tag{3}$$

- The above system has a solution $\mathcal{L} : TG \to \mathbb{R}$ if and only if there exists a function $\mathcal{L}_0 : \mathfrak{g} \to \mathbb{R}$ such that the following equations hold:

$$\alpha^i \frac{\partial \mathcal{L}_0}{\partial \alpha^i} = k\mathcal{L}_0, \tag{4}$$

$$[a, \alpha]^i \frac{\partial \mathcal{L}_0}{\partial \alpha^i} = 0, \quad \forall a \in \mathfrak{g}. \tag{5}$$

- For every $k \neq 0$, the function $\mathcal{L}_0 : \mathfrak{g} \to \mathbb{R}$ is a solution of the equation (5) if and only if $\mathcal{L}_0^k : \mathfrak{g} \to \mathbb{R}$ is also a solution of (5).

Theorem

The canonical spray S of a Lie group is left invariant projectively Riemann (resp. Finsler) metrizable if and only if it is left invariant Riemann (resp. Finsler) metrizable.

Proof.

„$\Rightarrow$" is trivial.

„$\Leftarrow$" If S is projectively Riemann (Finsler) metrizable, then there exist a left invariant quadratic (2-homogeneous) $\overline{E} : TG \to \mathbb{R}$ such that $\left(\frac{\partial^2 \overline{E}}{\partial \alpha^i \partial \alpha^j}\right)$ is positive definite and $\overline{F} = \sqrt{2\overline{E}}$ satisfies the Euler-Lagrange PDE. That is, $\overline{F}$ is a solution of $(1), (2), (3)$ with $k = 1$. Then $\overline{F}_0 = \overline{F}|_{\mathfrak{g}}$ satisfies $(4), (5)$. Furthermore $\overline{E}_0 = \frac{1}{2}(\overline{F}_0)^2$ is a solution of (5) and satisfies (4) with $k = 2$. The trivial extension of $\overline{E}_0$ is an energy function of the Riemann (Finsler) metric associated to S. □

Theorem

The canonical spray of a Lie group is left invariant projectively Finsler metrizable if and only if it is left invariant Riemann metrizable.

Proof.

„$\Rightarrow$" Riemann metrizability $\Rightarrow$ Finsler metrizability $\Rightarrow$ proj. Finsler metrizability.
„$\Leftarrow$" If S is projectively Finsler metrizable, then because of the above theorem it is Finsler metrizable. Since the canonical S is quadratic $\Rightarrow$ the associated connection is linear and the Finsler metric is a Berwald metric. Due to the Szabó's theorem to every Berwald metric $\widetilde{g}$ there exist a Riemann metric g such that $\widetilde{g}$ and g have the same geodesics and this completes the proof. $\square$

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Thank you for your attention.