The Flow Semigroup: An Algebraic Invariant for Graphs and Digraphs, and Some Results on Its Structure

Chrystopher L. Nehaniv,
John Rhodes, Gábor Horváth, Attila Egri-Nagy,
Károly Podoski, Paolo Dini, Piet Van Mieghem

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Wolfson Royal Society BioComputation Lab,
School of Computer Science, University of Hertfordshire, UK

SEVENTH FRAMEWORK PROGRAMME

Graphs and Digraphs

- *Digraph*: $\Gamma = (V, E)$, with *vertices* V, *edges* $E \subseteq V \times V$.
- The *reverse edge* to $e = (x, y) \in E$ is $\bar{e} = (y, x)$
- *(Undirected) Graph*: E is symmetric
- No self-loops: $(x, x) \notin E$.

BIOMICS

Spectrum of Graph

Algebraic graph theory

- *adjacency matrix* $A = A(\Gamma)$ of $\Gamma = (V, E)$ is a matrix of 0's and 1's with $A_{x,y} = 1$ iff $(x, y) \in E$.
- The multiset of eigenvalues of A is called the *spectrum* of A.

Theorem

The spectrum of graphs and digraphs is invariant for isomorphic (di)graphs.

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Example (Co-spectral Graphs)

$K_{1,4}$ and $C_4 \sqcup K_1$ have same spectrum. $\Longrightarrow$ not complete invariant

Elementary Collapsings of Directed Edges

Digraph $\Gamma = (V, E)$

Elementary collapsing

edge $e = (x, y) \longrightarrow$ function $T_{x,y} \colon V \to V$

$$T_{x,y}(v) = \begin{cases} y & \text{if } v = x, \\ v & \text{otherwise.} \end{cases}$$

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Example

$$\begin{aligned} T_{(1,2)} : 1 &\to 2 \\ 2 &\to 2 \\ 3 &\to 3 \\ 4 &\to 4 \end{aligned}$$

Defect k mappings

Definition

We say a mapping $f : V \to V$ has *defect k* if $|f(V)| = |V| - k$.

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f has defect $k \implies fg$ and gf have defect at least k.

Composition of Elementary Collapsings

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compose functions from left to right

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Definition

The *transformation semigroup* of a digraph $\Gamma = (V, E)$ to be the collection of transformations of V generated by the T_e $(e \in E)$.

$$S(\Gamma) = \langle T_{x,y} \in V^V \,|\, (x, y) \text{ is an edge of } \Gamma \rangle.$$

This is also called the semigroup of flows on Γ.

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$S(\Gamma)$ is an invariant for graphs and for digraphs.

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Main Theorem (Nehaniv–Rhodes)

$S(\Gamma)$ is a complete algebraic invariant for undirected graphs.

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Now $f^2 = T_{1,3}$, corresponding to an edge $(1,3)$.

Complete Invariance for Graphs

Lemma

If $T_{x,y} = T_{x_1,y_1} \cdots T_{x_k,y_k}$, *then* $T_{x,y}$ *or* $T_{y,x}$ *appears among the* T_{x_i,y_i}. *In fact, it appears as* T_{x_1,y_1}.

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Corollary

Let Γ be a digraph.
If $T_{x,y} \in S(\Gamma)$ then either $e = (x, y)$ or $\bar{e} = (y, x)$ is an edge of Γ.

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$\Longrightarrow$ For any graph Γ, we can recover its edges (and only these) from its flow semigroup.
This proves the Main Theorem, i.e. $S(\Gamma)$ is a complete invariant for graphs.

Reversing Edges in Directed Cycles

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W defect 1: permutes $n-1$ vertices cyclically

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Complete Invariance for Graphs

Caveat

NB: It is the transformation semigroup that is the complete invariant for graphs, not just the semigroup.

Example

For any graph Γ, the graph $\Gamma \sqcup K_1$ (new isolated vertex) has isomorphic semigroup.
Their transformation semigroups are not isomorphic, since one acts on $|V|$ points and the other on $|V| + 1$ points.

Induced Subgraphs

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Defect k Permutator Groups

Permutator Group

- If $e^2 = e \colon V \to V$ is an idempotent $S(\Gamma)$.
  Let $X_e = V \cdot e = \{v \cdot e \,|\, v \in V\}$
  Let $G_e =$ (unique) maximal subgroup of $S(\Gamma)$ containing e.
- (X_e, G_e) (faithful) *permutator group* of subset X_e, consists of defect k where $k = |V| - |X_e|$.

Theorem

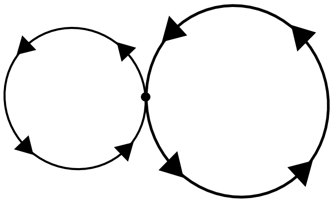
The defect k permutator groups (up to isomorphism) are invariants for digraphs.

Some Defect 1 Permutator Groups

Example

- Cycle graph with n nodes: Defect 1 group: cyclic C_{n-1}

Some Defect 1 Permutator Groups



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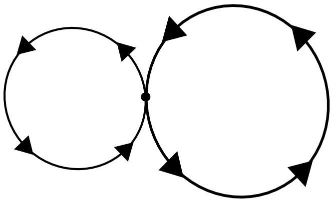

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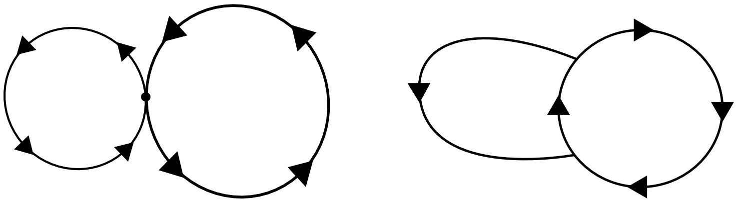

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- Overlapping Cycles:

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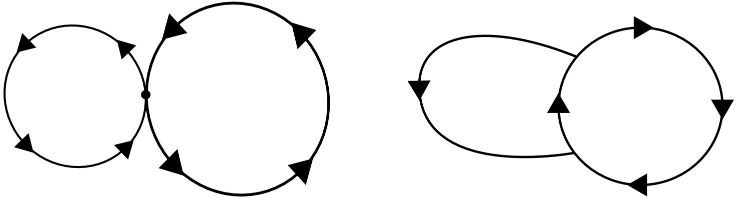

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- Two cycle graphs n and m nodes: Defect 1 group: $C_{n-1} \times C_{m-1}$
- Overlapping Cycles: Defect 1 group: symmetric or alternating

Defect k Groups of a Graph Γ

- unions of disjoint cycles $\rightarrow$ permutations
- maximal subgroups $=$ direct product of groups for strongly connected components.
- enough to look at strongly connected digraphs
- *bridge* is an undirected edge connecting two components.
- n-edge connected graph : removing any $n-1$ edges does not disconnect the graph
- no bridges $=$ 2-edge connected $\Longrightarrow$ Rhodes conjecture holds (Main Theorem II)

Main Theorem II (Egri-Nagy, Horváth, Nehaniv, Podoski, Rhodes, 2015)

Graph Γ, 2-edge connected, not a cycle.
Then maximal defect k groups are all isomorphic and

- Defect 1: Direct product of cyclic, alternating and symmetric groups of various orders.
- Defect 2: A_{n-2} or S_{n-2}
- Defect k : S_{n-k}

with finitely many exceptions.

Finding the Defect k Permutator Groups

Definition (The Defect k Digraph of a Digraph Γ, $1 \leq k \leq |V| - 2$)

D_k digraph :

nodes: $V \cdot f = f(V)$ with $f^2 = f \in S(\Gamma)$ having defect k.

edges: e in Γ labels a edge from X to Y in D_k iff $X \cdot T_e = Y$.

That is, T_e maps the elements of X one-to-one onto Y.

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Theorem (Rhodes)

Let $e^2 = e \in S(\Gamma)$. The permutator group G_e is generated by all directed cycles in D_k starting at X_e.

Defect k Groups in Particular Graphs

Example (Graph Δ : simple closed walk with n nodes)

- C_{n-1} : permutation group of defect 1 in $S(\Gamma)$
- Moreover, C_{n-k} : permutation group of defect k in $S(\Gamma)$ acting on successively smaller state sets for $k = 1, 2, ..., n-2$.

Example (Δ plus one new, non-reversed edge (Jordan's Lemma))

defect 1: alternating group A_{n-1} or symmetric group S_{n-1}
defect k: symmetric goup S_{n-k} on $|V| - k$ nodes ($k = 2, ..., n-2$).

Example (Surprise: $K_{1,n}$ tree, $n \geq 3$)

Defect 1 groups: trivial. Defect $k \geq 2$: $S(K_{1,n})$ has nontrivial subgroups, generated by 'swap' operations using temporary storage. Symmetric group S_{n+1-k}

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Open Problems

- If the defect k groups of graphs Γ and Γ' are known, what are they when we join Γ and Γ' by a bridge edge? or by a sequence of bridge edges?
- What are the defect k groups for connected Γ when Γ is not 2-connected?

Acknowledgment

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Induced Subgraphs and Applications

Theorem

If Γ' is a subgraph of Γ, then $S(\Gamma')$ is a sub- transformation semigroup of $S(\Gamma)$.

cf. Interlacing theorem for the spectrum of graphs.

Biochemical Reactions

Biochemical transitions are modelled as products of commuting elementary collapsings, $f = \prod T_{a,b}$, where $T_{x,y}$ and $T_{y,z}$ do not both occur among the $T_{a,b}$ for any x,, y, and z.
Notice that these are idempotents.
Any transformation semigroup generated by these is obviously a sub-transformation semigroup of the flow semigroup of the underlying digraph.

Krohn-Rhodes Theory and the Flow Semigroup $S(\Gamma)$

Theorem

Let Γ be a digraph with n nodes. Then
$S(\Gamma)$ divides a cascade of $H_{n-2} \wr \cdots \wr H_1$,
where H_i is a direct product (or disjoint group-product) of defect i groups with constant maps adjoined.

(One needs permutator groups in H_k per strongly connected component of the D_k.)
Proof: Apply the holonomy decomposition theorem to $S(\Gamma)$ □

Remark. Γ is connected $\Longrightarrow$ at most one defect k group occurs at each level.

Corollary

Krohn-Rhodes complexity of $S(\Gamma)$ is at most $|V| - 2$ for $|V|$ node (di)graphs.

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