How can we find ordinary differential equations having a given symmetry group?

Ágota Figula

University of Debrecen

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Preliminaries

Symmetries of a differential equation are transformations that move continuously a solution of the equation into another solution. Thus for each symmetry there exists a corresponding vector field (the *infinitesimal generator* of the symmetry). In the case of ordinary differential equations of order m the space of the variables $x, y, y', \cdots, y^{(m)}$ is called the jet space. The differential equation $f(x, y, y', \cdots, y^{(m)}) = 0$ defines an $m+1$-dimensional surface in this space which is called the hull of the differential equation. A smooth solution is a continuously differentiable function $\varphi(x)$ such that the curve $y = \varphi(x)$ with $y' = \frac{d\varphi(x)}{dx}, \cdots, y^{(m)} = \frac{d^m\varphi(x)}{dx^m}$ belongs to the hull, that is, $f(x, \varphi(x), \cdots, \frac{d^m\varphi(x)}{dx^m}) = 0$ identically for all x holds.

Preliminaries

The symmetries of a differential equation form a group, which is called the symmetry group. It is the group of transformations of the (x, y)-plane the prolongation of which to the derivatives $y', \cdots, y^{(m)}$ leaves the hull of the equation under consideration invariant. The symmetries of a differential equation one can use for generating from a known solution of the differential equation a new solution and creating new methods for solving it, for example lowering the order of the differential equation. In the process of integrating differential equations the important step is the simplification of the hull by a suitable change of variables.

Lie's method

Lie has determined the groups of transformations of the (x, y)-plane and written these into canonical form. He provided a classification of all ordinary differential equations of arbitrary order which admit these given groups as groups of their symmetries. We wish to present his method and give examples.

Let G be a given r-dimensional real Lie group of transformation of the (x, y)-plane. It is given by the basis elements (the infinitesimal generators) of its tangential Lie algebra $\mathbf{g}$:

$$
\begin{array}{rcl}
X_1 & = & \phi_1(x, y)\frac{\partial}{\partial x} + \eta_1(x, y)\frac{\partial}{\partial y}, \\
X_2 & = & \phi_2(x, y)\frac{\partial}{\partial x} + \eta_2(x, y)\frac{\partial}{\partial y}, \\
\dots & & \dots \\
X_r & = & \phi_r(x, y)\frac{\partial}{\partial x} + \eta_r(x, y)\frac{\partial}{\partial y}.
\end{array}
\tag{1}
$$

The m-th prolonged vector fields $X_i^{(m)}$, $i = 1, 2, \cdots, r$, are defined as

$$X_i^{(m)} = \phi_i(x,y)\frac{\partial}{\partial x} + \eta_i(x,y)\frac{\partial}{\partial y} + \eta_i^{(1)}(x,y,y')\frac{\partial}{\partial y'} +$$

$$+ \cdots + \eta_i^{(m)}(x,y,\cdots,y^{(m)})\frac{\partial}{\partial y^{(m)}}, \tag{2}$$

where $\eta_i^{(k)}$, $k = 1, 2, ..., m$, is the k-th coordinate function of the prolongation of the vector field X_i. We define recursively the functions $\eta_i^{(k)}$ as follows:

$$\eta_i^{(k)} = \frac{d\eta_i^{(k-1)}}{dx} - y^{(k)}\frac{d\phi_i}{dx},$$

that is

(3)

$$\eta_i^{(1)}(x,y,y') = \frac{\partial \eta_i(x,y)}{\partial x} + \frac{\partial \eta_i(x,y)}{\partial y} y' - \frac{\partial \phi_i(x,y)}{\partial x} y' - \frac{\partial \phi_i(x,y)}{\partial y} (y')^2,$$

$$\eta_i^{(2)}(x,y,y',y^{(2)}) = \frac{\partial \eta_i^{(1)}(x,y,y')}{\partial x} + \frac{\partial \eta_i^{(1)}(x,y,y')}{\partial y} y' +$$

(4) $$\frac{\partial \eta_i^{(1)}(x,y,y')}{\partial y'} y^{(2)} - \frac{\partial \phi_i(x,y)}{\partial x} y^{(2)} - \frac{\partial \phi_i(x,y)}{\partial y} y' y^{(2)},$$

and so further.
The m-th prolonged vector fields $X_i^{(m)}$, $i = 1, 2, \cdots, r$, depend on x, y, y', $\cdots$, $y^{(m)}$ and they are generators of the Lie algebra $\mathbf{g}$.

A differential equation

$$f(x, y, y', \cdots, y^{(m)}) = 0$$

of order m admits a group of symmetries whose Lie algebra is $\mathbf{g}$ if and only if the following system of partial differential equations is satisfied:

$$(5) \qquad \begin{array}{rcl} \phi_1 \frac{\partial f}{\partial x} + \eta_1 \frac{\partial f}{\partial y} + \eta_1^{(1)} \frac{\partial f}{\partial y'} + \cdots + \eta_1^{(m)} \frac{\partial f}{\partial y^{(m)}} & = & 0 \\ \phi_2 \frac{\partial f}{\partial x} + \eta_2 \frac{\partial f}{\partial y} + \eta_2^{(1)} \frac{\partial f}{\partial y'} + \cdots + \eta_2^{(m)} \frac{\partial f}{\partial y^{(m)}} & = & 0 \\ \cdots & & \\ \phi_i \frac{\partial f}{\partial x} + \eta_i \frac{\partial f}{\partial y} + \eta_i^{(1)} \frac{\partial f}{\partial y'} + \cdots + \eta_i^{(m)} \frac{\partial f}{\partial y^{(m)}} & = & 0 \\ \cdots & & \\ \phi_r \frac{\partial f}{\partial x} + \eta_r \frac{\partial f}{\partial y} + \eta_r^{(1)} \frac{\partial f}{\partial y'} + \cdots + \eta_r^{(m)} \frac{\partial f}{\partial y^{(m)}} & = & 0 \end{array}$$

whenever $f(x, y, y', \cdots, y^{(m)}) = 0$ holds.

Let M be the matrix

$$M = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \dots & \phi_r \\ \eta_1 & \eta_2 & \eta_3 & \dots & \eta_r \\ \eta_1^{(1)} & \eta_2^{(1)} & \eta_3^{(1)} & \dots & \eta_r^{(1)} \\ & \dots & & & \\ \eta_1^{(m)} & \eta_2^{(m)} & \eta_3^{(m)} & \dots & \eta_r^{(m)} \end{pmatrix}. \tag{6}$$

Then the system of partial differential equations given by (5) can be treated as the following system of linear equations in the variables $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \dots, \frac{\partial f}{\partial y^{(m)}}$:

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial y'} & \dots & \frac{\partial f}{\partial y^{(m)}} \end{pmatrix} \cdot M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

The case $m = r - 2$

The coefficient matrix M is an $(m+2) \times r$ matrix. Thus the system (5) has more than only the trivial solution $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \ldots = \frac{\partial f}{\partial y^{(m)}} = 0$ if and only if the rank of M is strictly less than $(m+2)$: $rk(M) < m+2$. Now, $rk(M) \leq r$ always holds. Hence if $m > r-2$, then the rank condition is automatically satisfied.

First we consider the case $m = r-2$. In that case one can arrive at the possible differential equations in the following way: Then the matrix M is an $(m+2) \times (m+2)$-matrix. The system (5) has a non-trivial solution f if and only if the rank of M is $< m+2$. Hence the determinant $D = |M|$ of M, which is a polynomial function of x, y, $y^{(i)}$, $i = 1, 2, \ldots, m = r-2$, has to be 0. When D is not identically 0 as a polynomial, then D is a polynomial function of x, y, $y^{(i)}$, $i = 1, 2, \ldots, m = r-2$, which has to be 0 if a non-trivial differential equation f exists. Hence by factoring D we obtain the only possibilities for such f.

The case $m < r - 2$

Now, assume $m < r - 2$. Then the coefficient matrix M is an $(m+2) \times r$-matrix. To obtain a non-trivial solution of the system (5) it is necessary that $rk(M) < m + 2$. Hence the determinants of all $(m+2) \times (m+2)$ submatrices of M has to be 0. Again, these determinants are polynomials of the variables x, y, $y^{(i)}$, $i = 1, 2, ..., m < r - 2$, and therefore their common factors provide the only possibilities for non-trivial differential equations f admitting G.

Theorem

Finding the differential equations $f(x, y, y', ..., y^{(m)}) = 0$ *of order* m*, which admit a group of symmetries whose Lie algebra is a given* r*-dimensional real Lie algebra* $\mathbf{g}$ *such that* $m \leq r - 2$*, one has to build the matrix*

$$M = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \dots & \phi_r \\ \eta_1 & \eta_2 & \eta_3 & \dots & \eta_r \\ \eta_1^{(1)} & \eta_2^{(1)} & \eta_3^{(1)} & \dots & \eta_r^{(1)} \\ & \dots & & & \\ \eta_1^{(m)} & \eta_2^{(m)} & \eta_3^{(m)} & \dots & \eta_r^{(m)} \end{pmatrix} \tag{8}$$

and compute the greatest common divisor of all $(m+2) \times (m+2)$ *subdeterminants. The factors of this polynomial give the only possibilities for the sought differential equations, unless this polynomial is identically* 0.

Example 1:

The Lie algebra $\mathbf{g}_1 = \mathbf{sl}_2(\mathbb{R})$ is generated by the vector fields:

$$(9) \qquad X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \; X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \; X_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}.$$

By Theorem (1) to obtain all first order ordinary differential equations which are invariant under the symmetry group corresponding to $\mathbf{g}_1$ we have to determine the first prolonged vector fields $X_i^{(1)}$. Since one has

$$(10) \qquad (\phi_1, \phi_2, \phi_3) = (1, x, x^2), (\eta_1, \eta_2, \eta_3) = (1, y, y^2)$$

using formula (3) we obtain

$$(11) \qquad (\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}) = (0, 0, (2y - 2x)y').$$

Since the determinant $D = \begin{vmatrix} \phi_i \\ \eta_i \\ \eta_i^{(1)} \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 0 & 0 & (2y-2x)y' \end{vmatrix}$ equals $2(y-x)^2 y'$ one can see that the first order differential equation $y' = 0$ is invariant under the symmetry group corresponding to the Lie algebra $\mathbf{g}_1$ given by (9) (cf. Theorem 1).

The case $m > r - 2$

How we can find the ordinary differential equations of order m which allow a given Lie group of dimension r as the group G of their symmetries such that $m > r - 2$? As before, G is an r-dimensional Lie group of transformations acting on (x, y)-plane such that the Lie algebra $\mathbf{g}$ of G is given by the infinitesimal generators: X_i, $i = 1, 2, ..., r$ in (1). We consider the m-th prolongations $X_i^{(m)}$ of the vector field X_i, $i = 1, 2, ..., r$:

$$X_i^{(m)} = \phi_i(x,y)\frac{\partial}{\partial x} + \eta_i(x,y)\frac{\partial}{\partial y} + \eta_i^{(1)}(x,y,y')\frac{\partial}{\partial y'} +$$

$$+ \cdots + \eta_i^{(m)}(x,y,\cdots,y^{(m)})\frac{\partial}{\partial y^{(m)}}. \tag{12}$$

They act on the $(m+2)$-dimensional manifold $\{x, y, y', \cdots, y^{(m)}\}$.

Assume that the determinant D in Theorem (1) is not identically zero. Then the system of the partial differential equations given by

$$
(13)\qquad
\begin{aligned}
\phi_1 \frac{\partial f}{\partial x} + \eta_1 \frac{\partial f}{\partial y} + \eta_1^{(1)} \frac{\partial f}{\partial y'} + \cdots + \eta_1^{(m)} \frac{\partial f}{\partial y^{(m)}} &= 0 \\
\phi_2 \frac{\partial f}{\partial x} + \eta_2 \frac{\partial f}{\partial y} + \eta_2^{(1)} \frac{\partial f}{\partial y'} + \cdots + \eta_2^{(m)} \frac{\partial f}{\partial y^{(m)}} &= 0 \\
\cdots & \\
\phi_i \frac{\partial f}{\partial x} + \eta_i \frac{\partial f}{\partial y} + \eta_i^{(1)} \frac{\partial f}{\partial y'} + \cdots + \eta_i^{(m)} \frac{\partial f}{\partial y^{(m)}} &= 0 \\
\cdots & \\
\phi_r \frac{\partial f}{\partial x} + \eta_r \frac{\partial f}{\partial y} + \eta_r^{(1)} \frac{\partial f}{\partial y'} + \cdots + \eta_r^{(m)} \frac{\partial f}{\partial y^{(m)}} &= 0
\end{aligned}
$$

has $m + 2 - r$ common solutions.

Example 1 for $m = 2$

To determine all ordinary differential equations of order 2 which are invariant under the 3-dimensional group of symmetries corresponding to the Lie algebra $\mathbf{g}_1$ we need to compute the 2-th prolonged vector fields $X_i^{(2)}$, $i = 1, 2, 3$, of (9). Using (10) we obtain $(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}) = (0, 0, (2y - 2x)y')$,

$$(\eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}) = (0, -y^{(2)}, -2y' + 2(y')^2 - 4xy^{(2)} + 2yy^{(2)}), \tag{14}$$

This yields the following system of partial differential equations:

$$\begin{aligned} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} &= 0 \\ x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} - y^{(2)}\frac{\partial f}{\partial y^{(2)}} &= 0 \\ x^2\frac{\partial f}{\partial x} + y^2\frac{\partial f}{\partial y} + 2(y-x)y'\frac{\partial f}{\partial y'} + & \\ +2(yy^{(2)} - 2xy^{(2)} + (y')^2 - y')\frac{\partial f}{\partial y^{(2)}} &= 0. \end{aligned} \tag{15}$$

From the first equation of (15) it follows that f depends on $x - y$. Using this the second and third equation reduces to

$$
\begin{array}{rrcl}
(16) & (y-x)\frac{\partial f}{\partial y} - y^{(2)}\frac{\partial f}{\partial y^{(2)}} & = & 0 \\
 & 2(y-x)y'\frac{\partial f}{\partial y'} + (2y'^2 - 2y' + 3(y-x)y^{(2)})\frac{\partial f}{\partial y^{(2)}} & = & 0.
\end{array}
$$

By integration of the first equation we obtain that its solutions are generated by:

$$
(17) \qquad y', \ \alpha = (y-x)y^{(2)}.
$$

Taking these as new variables the vector field

$$Y = 2(y-x)y' \frac{\partial}{\partial y'} + (2y'(y'-1) + 3(y-x)y^{(2)}) \frac{\partial}{\partial y^{(2)}}$$

takes the form

$$(18) \quad \widetilde{Y} = Y(y') \frac{\partial}{\partial y'} + Y(\alpha) \frac{\partial}{\partial \alpha} = 2y' \frac{\partial}{\partial y'} + (3\alpha + 2y'^2 - 2y') \frac{\partial}{\partial \alpha}.$$

The solution of the partial differential equation

$$2y' \frac{\partial f}{\partial y'} + (3\alpha + 2y'^2 - 2y') \frac{\partial f}{\partial \alpha} = 0$$

is

$$\varphi_1 = \alpha y'^{-\frac{3}{2}} - 2(y'^{\frac{1}{2}} + y'^{-\frac{1}{2}}) =$$

$$(19) \qquad (y-x)y^{(2)} y'^{-\frac{3}{2}} - 2(y'^{\frac{1}{2}} + y'^{-\frac{1}{2}}).$$

The case $m + 2 = r + 1$

If $m + 2 = r + 1$, then there exists a solution φ_1 of (5) which can be found by integration. The solution φ_1 depends solely on the variables $x, y, y', \cdots, y^{(r-1)}$. It follows from the following: We have to solve the system of the partial differential equations given by

$$
(20) \qquad
\begin{array}{rcl}
\phi_1 \frac{\partial f}{\partial x} + \eta_1 \frac{\partial f}{\partial y} + \eta_1^{(1)} \frac{\partial f}{\partial y'} + \cdots + \eta_1^{(r-1)} \frac{\partial f}{\partial y^{(r-1)}} & = & 0 \\
\phi_2 \frac{\partial f}{\partial x} + \eta_2 \frac{\partial f}{\partial y} + \eta_2^{(1)} \frac{\partial f}{\partial y'} + \cdots + \eta_2^{(r-1)} \frac{\partial f}{\partial y^{(r-1)}} & = & 0 \\
\ldots & & \\
\phi_i \frac{\partial f}{\partial x} + \eta_i \frac{\partial f}{\partial y} + \eta_i^{(1)} \frac{\partial f}{\partial y'} + \cdots + \eta_i^{(r-1)} \frac{\partial f}{\partial y^{(r-1)}} & = & 0 \\
\ldots & & \\
\phi_r \frac{\partial f}{\partial x} + \eta_r \frac{\partial f}{\partial y} + \eta_r^{(1)} \frac{\partial f}{\partial y'} + \cdots + \eta_r^{(r-1)} \frac{\partial f}{\partial y^{(r-1)}} & = & 0.
\end{array}
$$

We have to determine the joint invariants of the $r-1$-th prolonged vector fields $X_i^{(r-1)}$, $i = 1, 2, \cdots, r-1$. To obtain these we determine by integration the functionally independent invariants $\alpha_j(x, y, ..., y^{(r-1)})$ of one of the vector fields, say $X_1^{(r-1)}$. These invariants are the solutions of the linear homogeneous partial differential equation

(21)

$$X_1^{(r-1)}(\alpha) = \phi_1 \frac{\partial \alpha}{\partial x} + \eta_1 \frac{\partial \alpha}{\partial y} + \eta_1^{(1)} \frac{\partial \alpha}{\partial y'} + \cdots + \eta_1^{(r-1)} \frac{\partial \alpha}{\partial y^{(r-1)}} = 0.$$

If $X_1^{(r-1)}(\alpha) \neq 0$ in a neighbourhood of a chosen point x_0, then there are r functionally independent invariants, hence r functionally independent solutions of the partial differential equation (21) in a neighbourhood of x_0.

Since any joint invariant φ must in particular be an invariant of $X_1^{(r-1)}$, we can write φ as some function of the computed invariants α_j, $j = 1, 2, ..., r$, of $X_1^{(r-1)}$. Using the invariants α_j, $j = 1, 2, ..., r$, of $X_1^{(r-1)}$ as new variables (coordinates), we express the remaining vector fields $X_2^{(r-1)}, \cdots, X_r^{(r-1)}$ in these new coordinates. Then we find joint invariants of these *new* $r-1$ vector fields $X_2^{(r-1)}(\alpha_j), \cdots, X_r^{(r-1)}(\alpha_j)$. This procedure works inductively leading eventually to the joint invariants of all the vector fields expressed in terms of the joint invariants of the first α_j, $j = 1, 2, ..., r$, of them. This gives the common solution $\varphi_1(\alpha_j) = \varphi_1(x, y, y', ..., y^{(r-1)})$ of (20).

If $m+2=r+2$, then there are two solutions of (5). One is φ_1, the other is φ_2 which depends on the variables $x, y, y', \cdots, y^{(r)}$. If $m+2=r+3$, then there are three solutions φ_i, $i=1,2,3$, such that φ_3 depends on x, y, $y', \cdots, y^{(r+1)}$. For arbitrary m there are $m+2-r$ solutions φ_j, $j=1,2,\cdots,m+2-r$. Now we show that it is enough to determine the solutions φ_1, φ_2 by integration. If we know the solutions φ_1, φ_2, then for all $i \geq 3$ the solutions φ_i can be found using φ_1 and φ_2 by differentiation in the following way: The equation

$$\varphi_2 - a\varphi_1 + b = 0$$

with arbitrary constants a, b is a differential equation of order r which is invariant under the action of $\mathbf{g}$.

By differentiation with respect to the variable x one gets that

$$\frac{d\varphi_2}{dx} - a\frac{d\varphi_1}{dx} = 0$$

or equivalently

$$\frac{\frac{d\varphi_2}{dx}}{\frac{d\varphi_1}{dx}} = a. \tag{22}$$

(22) is a differential equation of order $r+1$ which is invariant under the action of $\mathbf{g}$. Hence one can choose $\frac{d\varphi_2}{dx} : \frac{d\varphi_1}{dx}$ as φ_3 and further $\frac{d\varphi_3}{dx} : \frac{d\varphi_1}{dx}$ as φ_4.

Theorem

Any differential equation of order $m > r + 2$ which admits the symmetry group the Lie algebra of which is the r-dimensional real Lie algebra $\mathbf{g}$ given by

$$X_i = \phi_i(x, y)\frac{\partial}{\partial x} + \eta_i(x, y)\frac{\partial}{\partial y}, \; i = 1, 2, \cdots, r,$$

has the form

$$\Omega(\varphi_1, \varphi_2, \varphi_3, \cdots) = 0,$$

with some smooth function Ω.

For the concrete calculation of differential equations of order $m > r + 2$ which admit the symmetry group the Lie algebra of which is given by the vector fields: X_i, $i = 1, 2, \cdots, r$, we proceed in the following way:

1. We have to find two common solutions φ_1, φ_2 of the system of partial differential equations given by (5) with $m = r$ such that φ_1 is a function of $x, y, \cdots, y^{(r-1)}$ and φ_2 is a function of $x, y, \cdots, y^{(r)}$. We compute the r-th prolonged vector fields $X_i^{(r)}$, $i = 1, 2, \cdots, r$.

2. Then we have to determine the two joint invariants of all the r-th prolonged vector fields $X_i^{(r)}$, $i = 1, 2, \cdots, r$. To obtain these we determine by integration the functionally independent invariants $\alpha_j(x, y, ..., y^{(r)})$ of one of the vector fields, say $X_1^{(r)}$. These invariants are the solutions of the linear homogeneous partial differential equation

$$X_1^{(r)}(\beta) = \phi_1 \frac{\partial \beta}{\partial x} + \eta_1 \frac{\partial \beta}{\partial y} + \eta_1^{(1)} \frac{\partial \beta}{\partial y'} + \cdots + \eta_1^{(r)} \frac{\partial \beta}{\partial y^{(r)}} = 0. \tag{23}$$

If $X_1(\beta) \neq 0$ in a neighbourhood of a chosen point x_0, then there are $r+1$ functionally independent invariants, hence $r+1$ functionally independent solutions of the partial differential equation (23) in a neighbourhood of x_0.

3. Since any joint invariant φ must in particular be an invariant of $X_1^{(r)}$, we can write φ as some function of the computed invariants β_j, $j = 1, 2, ..., r+1$, of $X_1^{(r)}$. Using the invariants β_j, $j = 1, 2, ..., r+1$, of $X_1^{(r)}$ as new variables (coordinates), we express the remaining vector fields $X_2^{(r)}, \cdots, X_r^{(r)}$ in these new coordinates.
4. Then we find joint invariants of these *new* $r-1$ vector fields $X_2^{(r)}(\beta_j), \cdots, X_r^{(r)}(\beta_j)$. This procedure works inductively leading eventually to the joint invariants of all the vector fields expressed in terms of the joint invariants of the first β_j, $j = 1, 2, ..., r+1$, of them.

Denote by K the subset of the set of indexes $\{1, 2, ..., r+1\}$ such that the invariants β_k does not depend on the variable $y^{(r)}$. The invariant φ_1 is a smooth function of these invariants β_k, $k \in K$, i.e. $\varphi_1 = \Psi_1(\beta_{k_1}, \beta_{k_2}, ...)$, $k_i \in K$. In particular φ_1 is the joint invariant of all the $r-1$-th prolonged vector fields $X_i^{(r-1)}$, $i = 1, 2, \cdots, r$. Therefore one has $\varphi_2 = \Psi_2(\varphi_1, \gamma_{l_1}, \gamma_{l_2}, ...)$, where $l_j \in \{1, 2, ..., r+1\} \setminus K$-s are the indexes of the invariants which depend on $y^{(r)}$.

5. We obtain the further common solutions φ_i, $i = 3, 4, \cdots$, of the system of partial differential equations (5) with arbitrary m, when we use the rule $\varphi_i = \frac{d\varphi_{i-1}}{dx} : \frac{d\varphi_1}{dx}$.

Example 1:

To determine all ordinary differential equations of order ≥ 2 which are invariant under the 3-dimensional group of symmetries corresponding to the Lie algebra $\mathbf{g}_1$ we need to compute the 3-th prolonged vector fields $X_i^{(3)}$, $i = 1, 2, 3$, of (9). Using (10) we obtain $(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}) = (0, 0, (2y - 2x)y')$,

(24) $(\eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}) = (0, -y^{(2)}, -2y' + 2(y')^2 - 4xy^{(2)} + 2yy^{(2)})$,

(25)
$(\eta_1^{(3)}, \eta_2^{(3)}, \eta_3^{(3)}) = (0, -2y^{(3)}, -6y^{(2)} + 6y^{(2)}y' - 6xy^{(3)} + 2yy^{(3)})$.

This yields the following system of partial differential equations:
(26)

$$\begin{aligned} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} &= 0 \\ x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} - y^{(2)}\frac{\partial f}{\partial y^{(2)}} - 2y^{(3)}\frac{\partial f}{\partial y^{(3)}} &= 0 \\ x^2\frac{\partial f}{\partial x} + y^2\frac{\partial f}{\partial y} + & \\ 2(y-x)y'\frac{\partial f}{\partial y'} + 2(yy^{(2)} - 2xy^{(2)} + (y')^2 - y')\frac{\partial f}{\partial y^{(2)}} + & \\ 2(yy^{(3)} - 3xy^{(3)} - 3y^{(2)} + 3y^{(2)}y')\frac{\partial f}{\partial y^{(3)}} &= 0. \end{aligned}$$

From the first equation of (26) it follows that f depends on $x - y$. Using this the second and third equation reduces to
(27)

$$\begin{aligned} (y-x)\frac{\partial f}{\partial y} - y^{(2)}\frac{\partial f}{\partial y^{(2)}} - 2y^{(3)}\frac{\partial f}{\partial y^{(3)}} &= 0 \\ 2(y-x)y'\frac{\partial f}{\partial y'} + (2y'^2 - 2y' + 3(y-x)y^{(2)})\frac{\partial f}{\partial y^{(2)}} + & \\ (6y^{(2)}y' - 6y^{(2)} + 4(y-x)y^{(3)})\frac{\partial f}{\partial y^{(3)}} &= 0. \end{aligned}$$

By integration of the first equation we obtain that its solutions are generated by:

$$y',\ \alpha = (y-x)y^{(2)},\ \beta = (y-x)^2 y^{(3)}. \tag{28}$$

Taking these as new variables the vector field

$$Y = 2(y-x)y'\frac{\partial}{\partial y'} + (2y'(y'-1) + 3(y-x)y^{(2)})\frac{\partial}{\partial y^{(2)}} +$$

$$+(6y^{(2)}(y'-1) + 4(y-x)y^{(3)})\frac{\partial}{\partial y^{(3)}}$$

takes the form

$$\widetilde{Y} = Y(y')\frac{\partial}{\partial y'} + Y(\alpha)\frac{\partial}{\partial \alpha} + Y(\beta)\frac{\partial}{\partial \beta} =$$

$$2y'\frac{\partial}{\partial y'} + (3\alpha + 2y'^2 - 2y')\frac{\partial}{\partial \alpha} + (4\beta + 6\alpha(y'-1))\frac{\partial}{\partial \beta}. \tag{29}$$

The solutions of the partial differential equation

$$2y'\frac{\partial f}{\partial y'} + (3\alpha + 2y'^2 - 2y')\frac{\partial f}{\partial \alpha} + (4\beta + 6\alpha(y'-1))\frac{\partial f}{\partial \beta} = 0$$

are generated by

$$\varphi_1 = \alpha y'^{-\frac{3}{2}} - 2(y'^{\frac{1}{2}} + y'^{-\frac{1}{2}}) =$$

(30) $$(y-x)y^{(2)}y'^{-\frac{3}{2}} - 2(y'^{\frac{1}{2}} + y'^{-\frac{1}{2}}),$$

$$\varphi_2 = \beta y'^{-2} - 6\varphi_1(y'^{\frac{1}{2}} + y'^{-\frac{1}{2}}) - 6(y' + y'^{-1}) =$$

(31)
$$(y-x)^2 y^{(3)} y'^{-2} - 6(y-x)y^{(2)}(y'^{-2} + y'^{-1}) + 18y' + 24 + 18y'^{-1}.$$

According to Theorem (2) the ordinary differential equations of order ≥ 2 which are invariant under the 3-dimensional group of symmetries corresponding to the Lie algebra $\mathbf{g}_1$ has the form

$$\Omega(\varphi_1, \varphi_2, \varphi_3, \cdots) = 0,$$

where φ_1, φ_2 has the form (30), (31) and for $i = 3, 4, \ldots$ one has $\varphi_i = \frac{d\varphi_{i-1}}{dx} : \frac{d\varphi_1}{dx}$.

The Lie algebra $\mathbf{g}_2 = \mathbf{sl_2}(\mathbb{R})$ is generated by the vector fields:

$$X_1 = \frac{\partial}{\partial x},\ X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},\ X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{32}$$

As one has

$$(\phi_1, \phi_2, \phi_3) = (1, 2x, x^2),\ (\eta_1, \eta_2, \eta_3) = (0, y, xy)$$

we obtain

$$(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}) = (0, -y', y - xy').$$

Hence the determinant $D = \begin{vmatrix} \phi_i \\ \eta_i \\ \eta_i^{(1)} \end{vmatrix} = \begin{vmatrix} 1 & 2x & x^2 \\ 0 & y & xy \\ 0 & -y' & y - xy' \end{vmatrix}$ equals y^2. So by Theorem 1 there does not exist any differential equation of order 1 which is invariant under the group of symmetries corresponding to the Lie algebra $\mathbf{g}_2$ given by (32).

To determine all ordinary differential equations of order ≥ 2 which are invariant under the 3-dimensional group of symmetries corresponding to the Lie algebra $\mathbf{g}_2$ we need to compute the 3-th prolonged vector fields $X_i^{(3)}$, $i = 1, 2, 3$, of (32). Using (33) we obtain $(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}) = (0, -y', y - xy')$,

$$(\eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}) = (0, -3y^{(2)}, -3xy^{(2)}),$$

$$(\eta_1^{(3)}, \eta_2^{(3)}, \eta_3^{(3)}) = (0, -5y^{(3)}, -3y^{(2)} - 5xy^{(3)}).$$

Therefore we have to solve the following system of partial differential equations:

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= 0 \\
2x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} - y'\frac{\partial f}{\partial y'} - 3y^{(2)}\frac{\partial f}{\partial y^{(2)}} - 5y^{(3)}\frac{\partial f}{\partial y^{(3)}} &= 0 \\
x^2\frac{\partial f}{\partial x} + xy\frac{\partial f}{\partial y} + (y - xy')\frac{\partial f}{\partial y'} - & \\
-3xy^{(2)}\frac{\partial f}{\partial y^{(2)}} - (3y^{(2)} + 5xy^{(3)})\frac{\partial f}{\partial y^{(3)}} &= 0.
\end{aligned}
\tag{33}
$$

From the first equation of (33) it follows that f does not depend on the variable x. Using this, system (33) reduces to the following:

$$
\begin{aligned}
y\frac{\partial f}{\partial y} - y'\frac{\partial f}{\partial y'} - 3y^{(2)}\frac{\partial f}{\partial y^{(2)}} - 5y^{(3)}\frac{\partial f}{\partial y^{(3)}} &= 0 \\
y\frac{\partial f}{\partial y'} - 3y^{(2)}\frac{\partial f}{\partial y^{(3)}} &= 0.
\end{aligned}
\tag{34}
$$

The solutions of the second equation are generated by

$$y,\ y^{(2)},\ \beta = \frac{3y^{(2)}y'}{y} + y^{(3)}. \tag{35}$$

Taking the functions in (35) as new variables the vector field $Y = y\frac{\partial}{\partial y} - y'\frac{\partial}{\partial y'} - 3y^{(2)}\frac{\partial}{\partial y^{(2)}} - 5y^{(3)}\frac{\partial}{\partial y^{(3)}}$ can be written into the form:

$$\widetilde{Y} = Y(y)\frac{\partial}{\partial y} + Y(y^{(2)})\frac{\partial}{\partial y^{(2)}} + Y(\beta)\frac{\partial}{\partial \beta} =$$

$$= y\frac{\partial}{\partial y} - 3y^{(2)}\frac{\partial}{\partial y^{(2)}} - 5\beta\frac{\partial}{\partial \beta}.$$

Using integration, the solutions of the partial differential equation

$$y\frac{\partial f}{\partial y} - 3y^{(2)}\frac{\partial f}{\partial y^{(2)}} - 5\beta\frac{\partial f}{\partial \beta} = 0$$

are generated by

$$\varphi_1 = y^3 y^{(2)}, \ \varphi_2 = \beta y^5 = y^4(3y^{(2)}y' + y^{(3)}y). \tag{36}$$

According to Theorem 2 any ordinary differential equation of order $m \geq 2$ which is invariant under the action of $\mathbf{g}_2$ has the form:

$$\Omega(\varphi_1, \varphi_2, \varphi_3, \cdots) = 0,$$

where φ_1, φ_2 is given by (36).

The Lie algebra $\mathbf{g}_3 = \mathbf{sl}_2(\mathbb{R})$ is generated by the vector fields:

$$X_1 = \frac{\partial}{\partial y},\ X_2 = y\frac{\partial}{\partial y},\ X_3 = y^2\frac{\partial}{\partial y}. \tag{37}$$

Because one has

$$(\phi_1, \phi_2, \phi_3) = (0,0,0),\ (\eta_1, \eta_2, \eta_3) = (1, y, y^2), \tag{38}$$

Using (3) we obtain $(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}) = (0, y', 2yy')$. In this case the determinant $D = \begin{vmatrix} \phi_i \\ \eta_i \\ \eta_i^{(1)} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 1 & y & y^2 \\ 0 & -y' & 2yy' \end{vmatrix}$ is identically 0.

As the coefficient of $\frac{\partial}{\partial y}$ in all vector fields given by (37) depends solely on y the differential equation $y' = 0$ is invariant under the group G corresponding to the Lie algebra $\mathbf{g}_3$.

To determine all ordinary differential equations of order ≥ 2 which are invariant under the 3-dimensional group of symmetries corresponding to the Lie algebra $\mathbf{g}_3$ we need to compute the 3-th prolonged vector fields $X_i^{(3)}$ of (37). Using (38) we get $(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}) = (0, y', 2yy')$,

$$(\eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}) = (0, y^{(2)}, 2(y')^2 + 2yy^{(2)}), \tag{39}$$

$$(\eta_1^{(3)}, \eta_2^{(3)}, \eta_3^{(3)}) = (0, y^{(3)}, 6y'y^{(2)} + 2yy^{(3)}). \tag{40}$$

Hence we obtain the following system of partial differential equations:

$$
\begin{aligned}
(41) \qquad \frac{\partial f}{\partial y} &= 0 \\
y\frac{\partial f}{\partial y} + y'\frac{\partial f}{\partial y'} + y^{(2)}\frac{\partial f}{\partial y^{(2)}} + y^{(3)}\frac{\partial f}{\partial y^{(3)}} &= 0 \\
y^2\frac{\partial f}{\partial y} + 2yy'\frac{\partial f}{\partial y'} + 2((y')^2 + yy^{(2)})\frac{\partial f}{\partial y^{(2)}} + & \\
+2(3y'y^{(2)} + yy^{(3)})\frac{\partial f}{\partial y^{(3)}} &= 0.
\end{aligned}
$$

From the first equation of (41) it follows that f does not depend on the variable y. Using this, system (41) reduces to the following:

$$
\begin{aligned}
(42) \qquad y'\frac{\partial f}{\partial y'} + y^{(2)}\frac{\partial f}{\partial y^{(2)}} + y^{(3)}\frac{\partial f}{\partial y^{(3)}} &= 0 \\
(y')^2\frac{\partial f}{\partial y^{(2)}} + 3y'y^{(2)}\frac{\partial f}{\partial y^{(3)}} &= 0.
\end{aligned}
$$

By integration of the second equation we obtain that its solutions are generated by:

$$x,\ y',\ \alpha = \frac{3}{2}(y^{(2)})^2 - y'y^{(3)}. \tag{43}$$

Taking the functions in (43) as new variables the vector field $X_2^{(3)} = y'\frac{\partial}{\partial y'} + \cdots + y^{(3)}\frac{\partial}{\partial y^{(3)}}$ can be written into the form:

$$X_2^{(3)}(y')\frac{\partial}{\partial y'} + X_2^{(3)}(\alpha)\frac{\partial}{\partial \alpha} = y'\frac{\partial}{\partial y'} + 2\alpha\frac{\partial}{\partial \alpha}. \tag{44}$$

Using integration, the solutions of the partial differential equation

$$y' \frac{\partial f}{\partial y'} + 2\alpha \frac{\partial f}{\partial \alpha} = 0$$

are generated by

$$\varphi_1 = x, \varphi_2 = \frac{\alpha}{(y')^2} = \frac{3(y^{(2)})^2 - 2y'y^{(3)}}{2(y')^2}.$$

Moreover, one has $\varphi_3 = \frac{d\varphi_2}{dx} = \frac{\beta}{(y')^3} = \frac{(y')^2 y^{(4)} + 3(y^{(2)})^3 - 4y'y^{(2)}y^{(3)}}{(y')^3}$ and so further. Therefore, any ordinary differential equation of order $m \geq 2$ which is invariant under the action of $\mathbf{g}_3$ has order at least 3 and takes the form:

$$\Omega(x, \varphi_2, \frac{d\varphi_2}{dx}, \frac{d^2\varphi_2}{dx^2}, \cdots) = 0,$$

where $\varphi_2 = \frac{3(y^{(2)})^2 - 2y'y^{(3)}}{2(y')^2}$ (cf. Theorem 2).

Lie algebra $\mathbf{g}_4 = \mathbf{sl_2}(\mathbb{R})$ is generated by the vector fields:

$$(45) \quad X_1 = \frac{\partial}{\partial x}, \; X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \; X_3 = (x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}.$$

This is the only one representation of the Lie algebra $\mathbf{sl_2}(\mathbb{R})$ such that the corresponding group action is primitive on the plane. Since one has

$$(46) \qquad (\phi_1, \phi_2, \phi_3) = (1, x, x^2 - y^2), (\eta_1, \eta_2, \eta_3) = (0, y, 2xy).$$

applying (3) we obtain

$$(47) \qquad (\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}) = (0, 0, 2y(1 + (y')^2).$$

Hence the determinant $D = \begin{vmatrix} \phi_i \\ \eta_i \\ \eta_i^{(1)} \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 - y^2 \\ 0 & y & 2xy \\ 0 & 0 & 2y(1+(y')^2) \end{vmatrix}$ equals $2y^2(1+(y')^2)$, according to Theorem 1 there does not exist any first order differential equation which admits a Lie group of symmetries having the Lie algebra $\mathbf{g}_4$ as its Lie algebra.

To determine all ordinary differential equations of order ≥ 2 which are invariant under the 3-dimensional group of symmetries corresponding to the Lie algebra $\mathbf{g}$ we need to compute the 3-th prolonged vector fields $X_i^{(3)}$ of (45). Applying (46) we obtain $(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}) = (0, 0, 2y(1 + (y')^2)$,

$$(\eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}) = (0, -y^{(2)}, 2y' + 2(y')^3 + 6yy'y^{(2)} - 2xy^{(2)})$$

$$(\eta_1^{(3)}, \eta_2^{(3)}, \eta_3^{(3)}) = (0, -2y^{(3)}, 12(y')^2y^{(2)} + 6y(y^{(2)})^2 + (8yy' - 4x)y^{(3)}).$$

Hence we have the following system of partial differential equations:

$$
(48)\qquad
\begin{aligned}
\frac{\partial f}{\partial x} &= 0\\
x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} - y^{(2)}\frac{\partial f}{\partial y^{(2)}} - 2y^{(3)}\frac{\partial f}{\partial y^{(3)}} &= 0\\
(x^2 - y^2)\frac{\partial f}{\partial x} + 2xy\frac{\partial f}{\partial y} + 2y(1+(y')^2)\frac{\partial f}{\partial y'} + &\\
2(y' + (y')^3 + 3yy'y^{(2)} - xy^{(2)})\frac{\partial f}{\partial y^{(2)}} + &\\
2(6(y')^2y^{(2)} + 3y(y^{(2)})^2 + (4yy' - 2x)y^{(3)})\frac{\partial f}{\partial y^{(3)}} &= 0.
\end{aligned}
$$

From the first equation of (48) it follows that f does not depend on the variable x. Using this, system (48) reduces to the following:

$$
(49)\qquad
\begin{aligned}
y\frac{\partial f}{\partial y} - y^{(2)}\frac{\partial f}{\partial y^{(2)}} - 2y^{(3)}\frac{\partial f}{\partial y^{(3)}} &= 0\\
2y(1+(y')^2)\frac{\partial f}{\partial y'} + 2(y' + (y')^3 + 3yy'y^{(2)})\frac{\partial f}{\partial y^{(2)}} + &\\
(12(y')^2y^{(2)} + 6y(y^{(2)})^2 + 8yy'y^{(3)})\frac{\partial f}{\partial y^{(3)}} &= 0.
\end{aligned}
$$

By integration of the first equation we obtain that its solutions are generated by:

$$y', \ \alpha = y^{(2)}y, \ \beta = y^{(3)}y^2. \tag{50}$$

Taking the functions in (50) as new variables the vector field $Y = 2y(1+(y')^2)\frac{\partial}{\partial y'} + 2(y' + (y')^3 + 3yy'y^{(2)})\frac{\partial}{\partial y^{(2)}} + (12(y')^2y^{(2)} + 6y(y^{(2)})^2 + 8yy'y^{(3)})\frac{\partial}{\partial y^{(3)}}$ can be written into the form:

$$\widetilde{Y} = (1+(y')^2)\frac{\partial}{\partial y'} + y'(1+(y')^2+3\alpha)\frac{\partial}{\partial \alpha} + (6(y')^2\alpha + 3\alpha^2 + 4y'\beta)\frac{\partial}{\partial \beta}. \tag{51}$$

Using integration, the solutions of the partial differential equation

$$(1+(y')^2)\frac{\partial f}{\partial y'}+y'(1+(y')^2+3\alpha)\frac{\partial f}{\partial \alpha}+(6(y')^2\alpha+3\alpha^2+4y'\beta)\frac{\partial f}{\partial \beta}=0$$

are generated by

$$\varphi_1 = \frac{\alpha+1+(y')^2}{(1+(y')^2)^{\frac{3}{2}}} = \frac{y^{(2)}y+1+(y')^2}{(1+(y')^2)^{\frac{3}{2}}},$$

(52)

$$\varphi_2 = \frac{\beta(y')^2+\beta-3y'\alpha^2}{(1+(y')^2)^3} = \frac{y^{(3)}y^2(1+(y')^2)-3y'(y^{(2)})^2y^2}{(1+(y')^2)^3}.$$

Hence any ordinary differential equation of order $m \geq 2$ which is invariant under the action of $\mathbf{g}_4$ takes the form:

$$\Omega(\varphi_1, \varphi_2, \cdots) = 0,$$

where φ_i, $i=1,2$, is given by (52) (cf. Theorem 2).

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