Base sizes, large orbits, and large character degrees of linear groups

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Base problem

For a permutation group $G \leq \mathrm{Sym}(\Omega)$ a base for G is a subset $X \subseteq \Omega$ whose pointwise stabilizer is the identity, i.e. for which $\cap_{x \in X} G_x = 1$. The concept of a base is fundamental both in the theory of permutation groups and in computational group theory.

The base size of a permutation group is the minimal number $b(G)$ such that a base of size $b(G)$ exists.

We were primarily interested in the base sizes of finite linear groups, that is, when V is a finite vector space and $G \leq GL(V)$ is a linear group with its natural action on V.

Several results were previously known for linear groups:

- If G is solvable and completely reducible, then $b(G) \leq 3$ (Á. Seress)
- If G is solvable and coprime (that is, $(|G|,|V|) = 1$), then $b(G) \leq 2$. (S. Dolfi, E. P. Vdovin)
- If G is any coprime linear group, then $b(G) \leq 94$. (D. Gluck, K. Magaard)

We were able to extend these results by showing the following.

- $b(G) \leq 2$ for any coprime linear group. [2]
- If V is a finite vector space over the q element field of characteristic p and $G \leq GL(V)$ is p-solvable with $O_p(G) = 1$, then $b(G) \leq 2$ unless $q \leq 4$ when $b(G) \leq 3$. [3]

On the proofs

We used an induction argument. In every step of the proof, by assuming that some bases are known for some subgroups or quotient groups with some induced actions, we were able to construct a base of the larger group.

Basically, this induction argument can be divided into the following steps:

- Reducing the problem to primitive linear groups;
- Analyzing tensor product actions;
- Solving the problem for almost quasisimple groups and for groups of symplectic type;
- A general induction argument which uses the results of the previous subcases.

We note that our proof strongly uses CFSG, especially a result of C. Köhler and H. Pahlings, who characterized all coprime almost quasisimple linear groups whith no regular orbit.

Large Orbits

For a linear group $G \leq GL(V)$, by a "large orbit" we mean a G-orbit in V of size at least $|G|^{1/2}$ or $|G|^{1/3}$. Thus, the existence of a large orbit is equivalent to the existence of a vector $v \in V$ with $|C_G(v)| \leq |G|^{1/2}$ or $|C_G(v)| \leq |G|^{2/3}$.

By using the above results about bases for linear groups we get the following

- If G is a coprime linear group, then there is a vector $v \in V$ with $|C_G(v)| \leq |G|^{1/2}$. (a question raised by I.M. Isaacs)
- If the characteristic of the base field of V is p, G is p-solvable and its action is completely reducible, then there is a vector $v \in V$ with $|C_G(v)| \leq |G|^{1/2}$ unless $64 \mid |V|$ or $81 \mid |V|$. In these cases we can guarantee the existence of a $v \in V$ with $|C_G(v)| \leq |G|^{2/3}$.

Gluck's conjecture

For a finite group G let

$$bc(G) = \max\{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$$

be the largest irreducible character degree of G and let $F(G)$ denote the Fitting subgroup of G.

A conjecture of Gluck asserts that $|G : F(G)| \leq bc(G)^2$ for every solvable group G.

Previous results of S. Dolfi and Y. Yang imply Gluck's conjecture unless $6 \mid |G|$.

If G is solvable, then $G/F(G)$ acts faithfully and completely reducibly on $F(G)/\Phi(G)$. This implies an action of $G/F(G)$ on $\mathrm{Irr}(F(G)/\Phi(G))$, which is again faithfull and completely reducible. Using our results on the existence of a large orbit with respect to this action we get a character $\lambda \in \mathrm{Irr}(F(G)/\Phi(G))$ such that the inequality $|G : I_G(\lambda)| \geq |G : F(G)|^{1/2}$ holds, which implies Gluck's conjecture by the Clifford correspondence. So we get

- If G is solvable and $|F(G)/\Phi(G)|$ not divisible by 64 nor 81 then Gluck's conjecture holds.
- Instead of an assumption on $|F(G)/\Phi(G)|$, we could show that Gluck's conjecture also holds if $F(G)/\Phi(G)$ is primitive as a G-module.

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Dropping solvability

Gluck has proved that there is an universal constant c such that $|G : F(G)| \leq bc(G)^c$ for every finite group G. In view of this result, it is a natural question what the best bounding constant c is. We managed to prove the following results in this direction.

- We have $|G : F(G)| \leq bc(G)^4$ for every finite group $|G|$.
- If G is a π-solvable group, where π is the set of the prime divisors of $|F^*(G)|$, then $|G : F(G)| \leq bc(G)^3$.
- If G is a finite non-abelian simple group, then $|G| \leq bc(G)^3$. (This result cannot be improved as the example $\mathrm{SL}(2, 2^f)$ shows.)
- The previous result can be used to prove that the product of the orders of the non-abelian composition factors of G is at most $bc(G)^3$. In particular, $|G : F(G)| \leq bc(G)^3$ if $G/F(G)$ has no abelian composition factor.

In view of the above results it is natural to ask the following question.

- Is it true that $|G : F(G)| \leq bc(G)^3$ for every finite group G?

Additional results

- If G is a group acting faithfully on a finite group K and $(|G|,|K|) = 1$ then there exist $x, y \in K$ such that $C_G(x) \cap C_G(y) = 1$.
- Let $G \leq \mathrm{Sym}(\Omega)$ with $t \nmid |G|$. Then there is a partition of Ω with at most t parts such that only the identity fixes this partition. (The so-called distinguishing number of G is at most t.)
- If V is a finite vector space over a field of characteristic p, $G \leq GL(V)$ is a p-solvable group with $O_p(G) = 1$, then $|G| \leq 24^{-1/3}|V|^d$ where $d = 3,243\ldots$. (This is an extension of a result of P.P. Pálfy and T.R. Wolf.)

References

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