On the Structure of Chevalley Groups

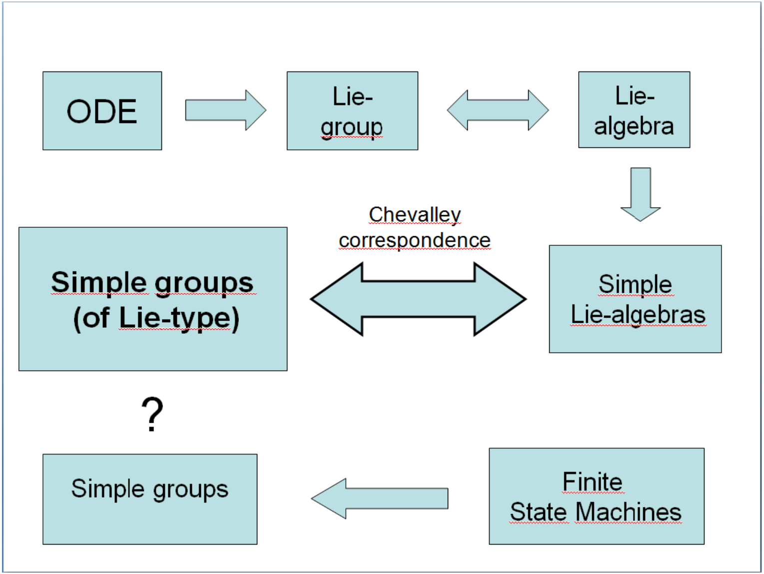
Károly Podoski

University of Debrecen

Second BIOMICS Summer Workshop
18-20 June 2014, University of St Andrews, Scotland

SEVENTH FRAMEWORK PROGRAMME

The Big Picture



Simple Lie algebras over complex field $\mathbb{C}$

- An I subalgebra of the Lie algebra L is called ideal if $[x, i] \in I$ for all $i \in I$ and $x \in L$.
- A Lie algebra L is simple if does not contain any non-trivial ideal.

  Let L be a **finite dimensional** simple Lie algebra over $\mathbb{C}$.

  Then

  there exist a unique (up to isomorphism) root space or Cartan decomposition such that

$$L = H \oplus \bigoplus_{r \in \Phi} L_r$$

  where H is a Cartan subalgebra and each root space L_r is one dimensional.
- The roots r can be represented as vectors of the **real** Euclidean space .
- $\dim H = |\Phi|$

Fundamental bases

There exist a fundamental base $\Pi \subset \Phi$ such that

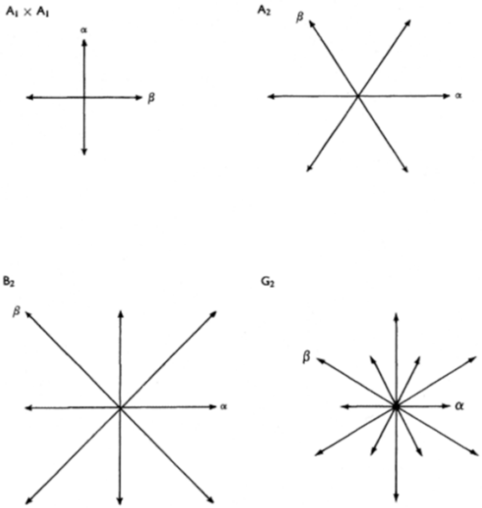
- the elements of Π are linearly independent.
- Every element of Φ is a linear combination of Π such that all multipliers are integer and all of them are positive or all of them are negative.
- $|\Pi| = l$ is called the rank of the root system Φ. This equals to the rank of the Lie algebra L.

There are only 4 cases of a root system rank of 2. Evenmore, for $\alpha, \beta \in \Phi$ $(\beta \neq \pm\alpha)$

$$\{\mathbb{C}\alpha + \mathbb{C}\beta\} \cap \Phi$$

is the one of the following four vector systems:

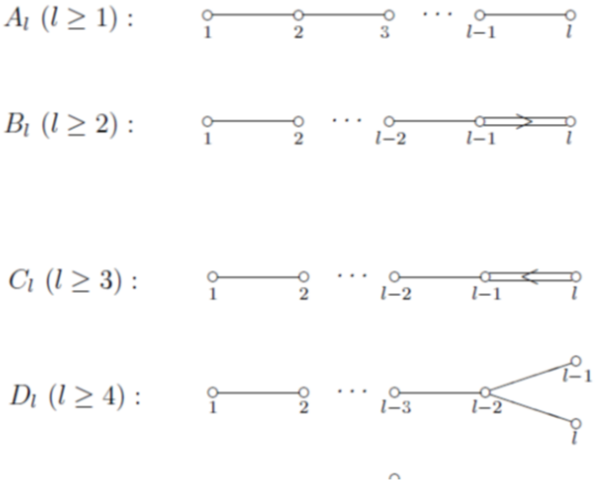
Root spaces of dimension 2



Dynkin diagrams

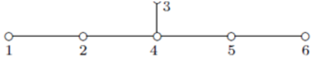
- Let Π be a fundamental base of elements l.
- The vertices of the Dynkin diagram are the elements of Π (Note that all fundamental bases are isomorphic) .
- Let $\alpha, \beta \in \Pi$ and θ be the angle between α and β. The number of edges between α and β is $4\cos^2\theta$.
- Why $4\cos^2\theta$ is integer?
- If $|\alpha| \neq |\beta|$ then there is an arrow on the edges directed to the shorter vector.

Infinite series of Dynkin diagrams

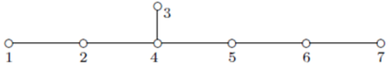


Infinite series of Dynkin diagrams

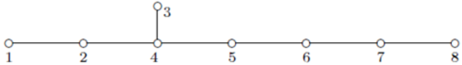
E_6 :



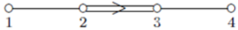
E_7 :



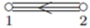
E_8 :



F_4 :



G_2 :



The structure constants of simple Lie algebras

Let L be a finite dimensional simple Lie algebra over $\mathbb{C}$. Take a Cartan decomposition such that

$$L = H \oplus \bigoplus_{r \in \Phi} L_r$$

Here one can choose elements $e_r \in L_r, e_{-r} \in L_{-r}, h_r \in H$ for each $r \in \Phi$ such that

- $[h_r, e_r] = 2e_r$, $[h_r, e_{-r}] = 2e_{-r}$ and $[e_r, e_{-r}] = h_r$. This means that for each $r \in \Phi$ the linear combinations of $e_r, e_{-r}, h_r \in H$ form a simple Lie subalgebra isomorphic to $\mathrm{sl}(2, \mathbb{C})$
- $[h_s, e_r] = A_{rs} e_r$ and $[e_r, e_s] = N_{rs} e_{r+s}$ $(r + s \neq 0)$. These constants A_{rs} and N_{rs} are called structure constants.

Simple Lie algebras over arbitrary field

Theorem (Chevalley)

The basis elements $e_r \in L_r, e_{-r} \in L_{-r}$ and $h_r \in H$ (for all $r \in \Phi$) can be choosen such that all structure constants A_{rs}, N_{rs} are integers.

Why this result is so essential?
Because one can define the simple Lie algebra $L = L(\mathbb{C})$ over $\mathbb{C}$ in an abstract way using the calculation rules:

Definition

- *Let $\{e_r, e_{-r}, h_r | r \in \Phi\}$ is a basis of L (over $\mathbb{C}$).*
- $[h_r, e_r] = 2e_r$, $[h_r, e_{-r}] = 2e_{-r}$ and $[e_r, e_{-r}] = h_r$.
- $[h_s, e_r] = A_{rs}e_r$ and $[e_r, e_s] = N_{rs}e_{r+s}$ $(r + s \neq 0)$.

Simple Lie algebras over arbitrary field

Since **all multipliers are integers** in the previous definition the complex field can change to an arbitrary other field K. Therefore the definition of the simple Lie algebra $L(K)$ over an arbitrary field K:

Definition

- *Let* $\{e_r, e_{-r}, h_r | r \in \Phi\}$ *is a basis of* $L(K)$ *(over* K*).*
- $[h_r, e_r] = 2e_r$, $[h_r, e_{-r}] = 2e_{-r}$ *and* $[e_r, e_{-r}] = h_r$.
- $[h_s, e_r] = A_{rs}e_r$ *and* $[e_r, e_s] = N_{rs}e_{r+s}$ $(r + s \neq 0)$.

The automorphisms of Lie algebras

- An isomorphism $\phi : L \to L$ is called automorphism of L
- All automorphisms of L form a group denoted by $\mathrm{Aut}(L)$
- The adjoint map $\mathrm{ad}x : L \to L$ such that $y \to [x, y]$ is a nilpotent derivation if $x = \lambda e_r$ for each $r \in \Phi$, $\lambda \in K$.
- If δ is a nilpotent derivation of L ($\delta^n = 0$) then the exponential map:

$$\exp(\delta) = \mathrm{id} + \delta + \frac{\delta^2}{2!} + \cdots + \frac{\delta^n}{n!}$$

is an automorphism of L

The automorphisms of Lie algebras

- An isomorphism $\phi : L \to L$ is called automorphism of L
- All automorphisms of L form a group denoted by $\mathrm{Aut}(L)$
- The adjoint map $\mathrm{ad}x : L \to L$ such that $y \to [x, y]$ is a nilpotent derivation if $x = \lambda e_r$ for each $r \in \Phi$, $\lambda \in K$.
- If δ is a nilpotent derivation of L ($\delta^n = 0$) then the exponential map:

$$\exp(\delta) = \mathrm{id} + \delta + \frac{\delta^2}{2!} + \cdots + \frac{\delta^n}{n!}$$

is an automorphism of L

The definition of Chevalley groups

Definition

The Chevalley group associated to the simple Lie algebra $L(K)$ is the $G_L(K)$ subgroup of $\mathrm{Aut}(L)$ such that

$$G_L(K) = < \exp(\mathrm{ad}\lambda e_r) | r \in \Phi, \lambda \in K >$$

- Depending on the choice of the field K ($K = \mathbb{R}$ or $K = \mathbb{Q}$) a Chevalley group can be infinite or finite (if K is a finite field)
- The Chevalley groups are simple (except a few cases) but the proof is not simple.

The structure of Chevalley groups

- Let $x_r(\lambda) = \exp(\mathrm{ad}\lambda e_r)$. The Chevalley group $G_L(K)$ is generated by all $x_r(\lambda)$, $r \in \Phi$, $\lambda \in K$.
- The root subgroup $X_r =< x_r(\lambda) | \lambda \in K >$
- The subgroup $< X_r, X_{-r} >$ is a homomorph image of $\mathrm{SL_2(K)}$ such that

$$\phi : \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \to x_r(\lambda)$$

$$\phi : \begin{pmatrix} 1 & 0 \\ \lambda & 0 \end{pmatrix} \to x_{-r}(\lambda)$$

Why is this important to find homorph images of $\mathrm{SL_2(K)}$ in the Chevalley group $G_L(K)$? Because, it is easy to calculate using matrices of 2×2.

(B,N)-pairs

- Let $h_r(\lambda) = \phi \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $n_r(\lambda) = \phi \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}$
- Let $B =< X_r | r \in \Phi >$
- Let $N =< n_r(\lambda), h_r(\lambda) | r \in \Phi, \lambda \in K >$
- In general if group has (B,N)-pair, so there exists subgroups B, N with nice properties then we have an argument to prove the simplicity of the group.

Properties (B,N)-pairs

Let B, N be subgroups of G

- G is generated by B and N
- $B \cap N$ is normal subgroup in N
- The group $W = N/B/B \cap N$ is generated by a set of element w_i $i \in I$ such that $w_i^2 = 1$
- If $n_i \in N$ maps to w_i under the natural homomorphism of N into W, and if n is any element of N, then

$$Bn_iB \cdot BnB \subseteq Bn_iNB \cup BnB.$$

- $n_iBn_i \neq B$