BIOMICS

The global version of the rectification theorem

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SEVENTH FRAMEWORK PROGRAMME

The main method for solving and studying differential equations is to choose a suitable change of variables, i.e., in geometric terms, a suitable diffeomorphism that simplifies the given vector field or direction field. Roughly speaking this is the so-called rectification theorem.

The main method for solving and studying differential equations is to choose a suitable change of variables, i.e., in geometric terms, a suitable diffeomorphism that simplifies the given vector field or direction field. Roughly speaking this is the so-called rectification theorem.

The ‘local’ version of this result is a well-known theorem in the field of ordinary differential equations. The main aim of this talk is to prove its ‘global’ counterpart. Therefore, firstly we will shortly recall the most important notions that will be used subsequently. While doing so, we will rely on the two basic monographs of this area Arnol’d [1] and Walter [5].

Definition

Let $M \subset \mathbb{R}^n$ be a manifold, a diffeomorphism is a one-to-one mapping $f\colon M \to N$ that is smooth as well as its inverse.

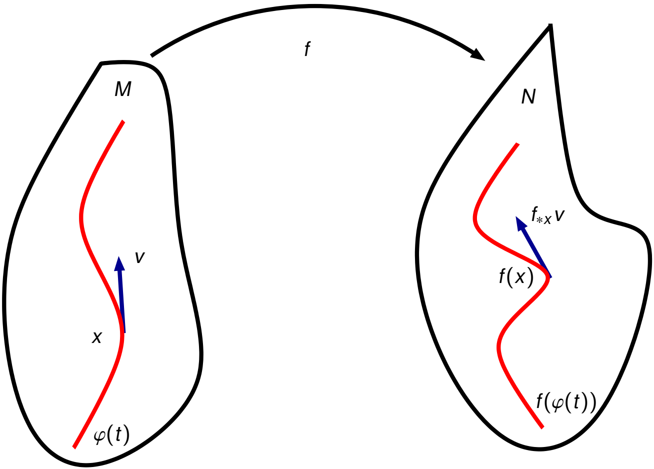
Definition

Let $f: M \to N$ be a smooth mapping of the domain M of a linear space into a domain N that is supposed to be a linear space and let v be a vector attached at the point $x \in M$. Then at the point $f(x) \in M$ also arises a vector denoted by $f_{*x}v$ called the image of the vector v under the mapping f and its is defined in the following

$$f_{*x}v = \frac{d}{dt} f\left(\varphi(t)\right)\Big|_{t=0},$$

where

$$\varphi(0) = x \quad \text{and} \quad \frac{d}{dt}\varphi(t)\Big|_{t=0} = v.$$



f
M
N
v
x
φ(t)
f∗x v
f(x)
f(φ(t))

Definition

The set of velocity vectors of motions leaving the point $x \in M$ is a linear space, called the tangent space to the domain M at the point $x \in M$ and its is denoted by $T_x M$.

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Proposition

The mapping $f_* \colon T_x M \to T_{f(x)} N$ is linear. Furthermore, if $f \colon M \subset \mathbb{R}^M \to \mathbb{R}^n$ and Cartesian coordinates have been chosen, then f_{*x} can be expressed in the following form

$$(f_{*x} v)_i = \sum_{j=1}^{m} \frac{\partial f_i}{\partial x_j} v_j \qquad (i = 1, \ldots, n)\,.$$

The linear mapping f_{*x} is called the derivative *of the function f at $x \in M$.*

Definition

We say that on the domain M a (smooth) vector field is given if for all $x \in M$ it is assigned a vector $v(x) \in T_x M$ such that the dependence from x is smooth. Assume that a smooth vector field v is given on the domain M, further let $f\colon M \to N$ be a diffeomorphism. Then the image of the vector field v under the diffeomorphism f is defined through

$$w(y) = (g_{*x})\, v(x) \qquad (y \in N)\,,$$

where $y = g(x)$.

Theorem

Let $f\colon M \to N$ be a diffeomorphism. Then the differential equation

$$\dot{x} = v(x) \qquad (x \in M) \tag{1}$$

with phase space M determined by the vector field v is equivalent to the equation

$$\dot{y} = (f_* v)\, y \qquad (y \in N) \tag{2}$$

with phase space N determined by the vector field $f_* v$. More precisely, a function $\varphi\colon I \to M$ is a solution to equation (1) is and only if $f \circ \varphi\colon I \to N$ is a solution to equation (2).

Definition

Let v be a vector field on M. A diffeomorphism $g\colon M \to M$ is called a symmetry *of v, in case g maps the vector field into itself, that is, if*

$$g_* v = v$$

is fulfilled. In such a situation, we say that the vector field v is invariant *under the symmetry g.*

The vector field v is termed to be invariant *with respect to a group of diffeomorphisms, if v is invariant under all transformations of this group.*

The integral curves of a field map into one another under the action of a symmetry and also conversely, if a diffeomorphism maps the integral curves of a direction field into one another, then this mapping is a symmetry.

Example

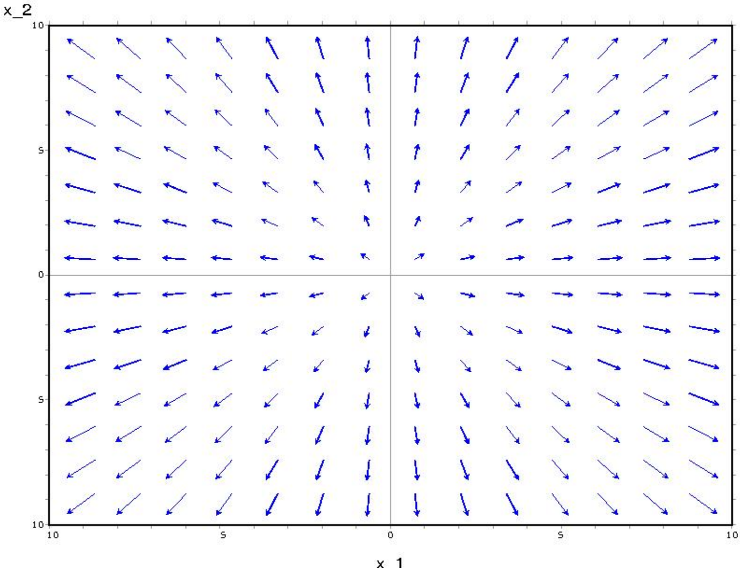
Let us consider on the plane the so-called Euler-field, that is, $x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$*. It is easy to see that this field admits all of the following symmetry groups*

(i) *the one-parameter group of dilatations*

$$x \to e^t x$$

(ii) *the one-parameter group of rotations through a fixed angle* t

(iii) *the one-parameter group of hyperbolic rotations*


x_2
10
5
0
5
10
10
5
0
5
10
x_1

Example

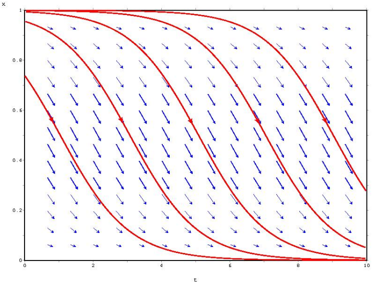
The direction field of the equation

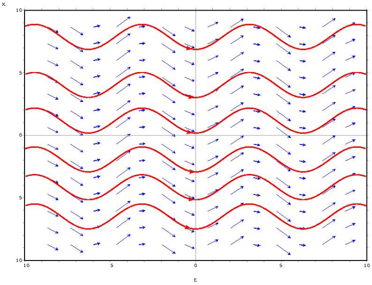
$$\dot{x}(t) = v(t)$$

in the extended phase space is invariant with respect to translations along the x*-axis, while the direction field of the equation*

$$\dot{x}(t) = v(x)$$

is invariant with respect to translations along the t*-axis.*


x
1
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10
t


x
10
5
0
5
10
10
5
0
5
10
t

Theorem (One-parameter symmetry group $\Rightarrow$ local integrability of the ODE)

Assume that a one-parameter symmetry group of a direction field defined on the plane is known. Then every non stationary point of the direction field has a neighborhood in which the differential equation determined by the direction field can be integrated explicitly. A point of the direction field is termed to be a *stationary point* *if it leaved fixed by all elements of the symmetry group.*

Theorem (Picard–Lindelöf)

Let $I \subset \mathbb{R}$ be a nonvoid interval, $v\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function and let us assume that there exists a continuous function $L\colon I \to [0, +\infty[$ such that

$$\left\| v(t, x_1) - v(t, x_2) \right\| \leq L(t)\, \|x_1 - x_2\| \quad (t \in I, x_1, x_2 \in \mathbb{R}^n)\,, \qquad (\mathcal{L})$$

Let $\xi \in I$, $\eta \in \mathbb{R}^n$, then the Cauchy problem

$$\begin{cases} \dot{x}(t) & = & v(t, x) \\ x(\xi) & = & \eta \end{cases}$$

admits exactly one $\varphi\colon I \to \mathbb{R}^n$ solution, any other solution is a restriction of this solution.

Theorem (Picard–Lindelöf (cont.))

Furthermore, for all continuous function $\varphi_0 \colon I \to \mathbb{R}^n$, the so-called *Picard iteration*, *i.e.,*

$$\varphi_{n+1}(t) = \eta + \int_{\xi}^{t} v\left(s, \varphi_n(s)\right) ds \quad (x \in I, n \geq 0)$$

converges pointwise to φ such that the convergence is uniform on every compact subinterval of I.

Theorem (Peano local existence theorem)

Let $D \subset \mathbb{R} \times \mathbb{R}^n$ be a nonempty and open set, $v \colon D \to \mathbb{R}^n$ be a continuous mapping. Then for every pair $(\xi, \eta) \in D$ the Cauchy problem

$$\begin{aligned} \dot{x}(t) &= v(t, x(t)) \\ x(\xi) &= \eta \end{aligned}$$

admits a solution in an appropriate neighborhood of (ξ, η).

Remark

Comparing the Picard–Lindelöf theorem with the Peano theorem we have the following. The Picard–Lindelöf theorem both assumes more and concludes more. It requires Lipschitz continuity, while the Peano theorem requires only continuity; but it proves both existence and uniqueness where the Peano theorem proves only the existence of solutions.

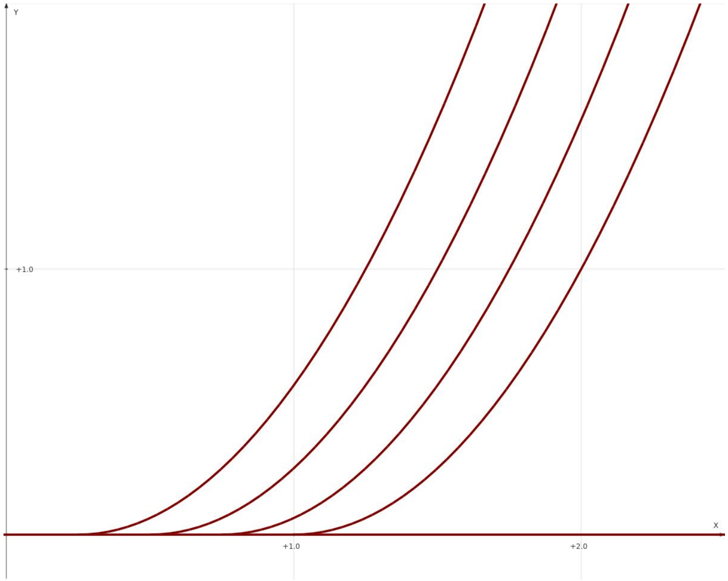
To illustrate the differences let us consider the following Cauchy problem

$$\begin{aligned} \dot{x}(t) &= 2\sqrt{|x|} \\ x(0) &= 0 \end{aligned}$$

It is easy to see that for any choices of $\alpha \geq 0$ the function

$$x_\alpha(t) = \begin{cases} 0, & t \leq \alpha \\ (t-\alpha)^2, & t \geq \alpha \end{cases}$$

solves the above initial value problem.


Y
+1.0
X
+1.0
+2.0

Thus, this Cauchy problem does not admits a unique solution, due to the fact that the mapping

$$\mathbb{R} \ni x \longmapsto 2\sqrt{|x|}$$

does not fulfill condition $(\mathcal{L})$. Indeed,

$$\sup_{x,y\in\mathbb{R},x\neq y} \frac{2\sqrt{|x|}-2\sqrt{|y|}}{x-y} \geq \sup_{x,y>0,x\neq y} \frac{2\sqrt{|x|}-2\sqrt{|y|}}{x-y} = \sup_{x,y>0,x\neq y} \frac{1}{\sqrt{x}+\sqrt{y}} = +\infty.$$

We also remark that even the Peano existence theorem can be generalized. Peano's theorem requires that the right-hand side of the differential equation is continuous, while Carathèodory's theorem shows existence of solutions (in a more general sense) for some discontinuous equations.

Definition

Let $I \subset \mathbb{R}$ be a nonvoid, open interval $D \subset I \times \mathbb{R}^n$ be a nonempty, open set and $v\colon D \to \mathbb{R}$ be a function, $(\xi, \eta) \in D$. We say that the function $x\colon J \to \mathbb{R}$ is a solution to the Cauchy problem

$$\begin{aligned} \dot{x}(t) &= v(t, x(t)) \\ x(\xi) &= \eta \end{aligned}$$

in the sense of Carathèodory, if

(i) *$J \subset I$ is an open interval,*

(ii) *$x\colon J \to \mathbb{R}^n$ is absolutely continuous on J,*

(iii) *for all $t \in J$ we have $(t, x(t)) \in D$,*

(iv) *for almost all $t \in J$*
$$\dot{x}(t) = v(t, x(t)).$$

Theorem (Carathèodory existence theorem)

Let $a, b \in \mathbb{R}$ be arbitrarily fixed, further $\xi \in \mathbb{R}$, $\eta \in \mathbb{R}^n$ and let

$$D = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \mid |t - \xi| \leq a, \ \|x - \eta\| \leq b\}.$$

Let us assume that the mapping $v \colon D \to \mathbb{R}$ is a Carathèodory function, i.e.,

(i) *for each fixed t the mapping*

$$x \longmapsto v(\cdot, x)$$

is continuous.

(ii) *for all x the function*

$$t \longmapsto v(t, \cdot)$$

is Lebesgue measurable.

(iii) *there is a Lebesgue integrable function μ such that*

$$\left\| v(t,x) \right\| \leq \mu(t) \qquad ((t,x) \in D).$$

Theorem (Carahèodory (cont.))

Then the initial value problem

$$\begin{aligned} \dot{x}(t) &= v(t, x) \\ x(\xi) &= \eta \end{aligned}$$

admits a solution (in the sense of Carathèodory) in an appropriate neighborhood of the point $(\xi, \eta) \in D$.

Remark

In view of the previous statement it is also possible to consider differential equations with discontinuous right hand side, such as

$$\begin{aligned} \dot{x}(t) &= H(t) \\ x(0) &= 0. \end{aligned}$$

Here, H denotes the so-called Heaviside function, that is,

$$H(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0. \end{cases}$$

The uniquely determined solution of the above Cauchy problem in the sense of Carathèodory is a 'ramp function', namely,

$$x(t) = \begin{cases} 0, & t \leq 0 \\ t, & t > 0 \end{cases} \qquad (t \in \mathbb{R})$$

Definition

Let $I \subset \mathbb{R}$ be a nonvoid, open interval $D \subset I \times \mathbb{R}^n$ be a nonempty, open set and $v: D \to \mathbb{R}$ be a function, $(\xi, \eta) \in D$. The solution $x: J \to \mathbb{R}$ of the Cauchy problem

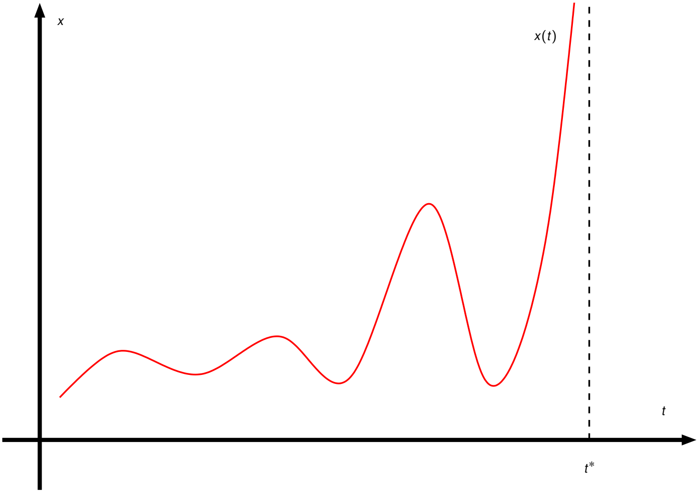
$$\begin{aligned} \dot{x}(t) &= v(t, x(t)) \\ x(\xi) &= \eta \end{aligned}$$

is said to blow up*, if there exists $t^* > 0$ such that for all $K > 0$ there exists $\varepsilon > 0$ so that in case $t < t^*$, $|t - t^*| < \varepsilon$, then*

$$\left\| x(t) \right\| > K$$

is fulfilled.

For further results concerning this notion see Ball [2] and Walter [5].


x
x(t)
t
t*

Remark

We would also like to remark that more regularity of the right hand side of the differential equation does not impose existence and uniqueness, even if the right hand side of the equation is analytic.

To see this, let us consider the following Cauchy problem

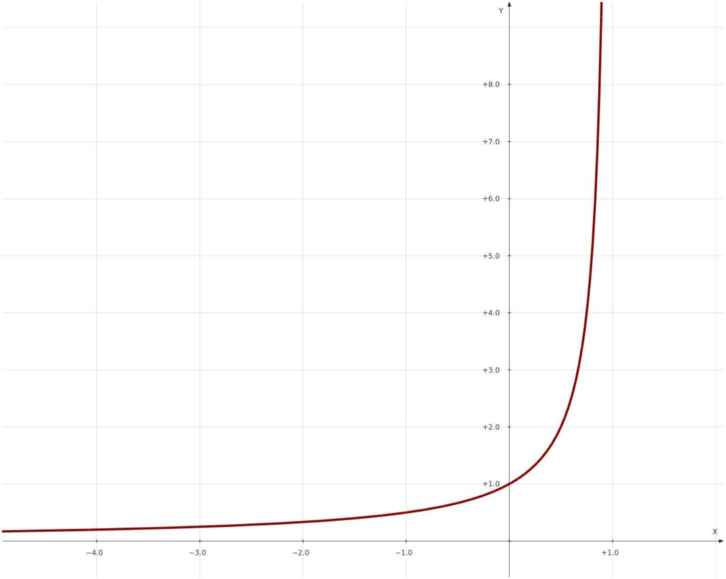
$$\begin{aligned} \dot{x}(t) &= x^2(t) \\ x(0) &= x_0. \end{aligned} \qquad (t \in \mathbb{R}).$$

Obviously, the mapping $v\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$v(t,x) = x^2 \qquad ((t,x) \in \mathbb{R} \times \mathbb{R})$$

is of the class $\mathcal{C}^\omega(\mathbb{R} \times \mathbb{R})$. Nevertheless, the above Cauchy problem has a unique maximal solution on $]-\infty, x_0^{-1}[$

$$x(t) = \frac{1}{\frac{1}{x_0} - t} \qquad \left(t \in]-\infty, x_0^{-1}]\right).$$


Y
+8.0
+7.0
+6.0
+5.0
+4.0
+3.0
+2.0
+1.0
X
−4.0
−3.0
−2.0
−1.0
+1.0

However, the Cauchy problem does not have a solution that is defined on the whole line $\mathbb{R}$. This phenomenon is caused by the fact that the differential equation in question has a singularity (in this case a movable singularity). In terms of the solutions, this yields that a blow up occurs near the singularity.

In view of the statements presented above, we can say that from one side the assumptions of the Picard–Lindelöf theorem are rather strong. On the other hand, the previous remarks show that with milder conditions we cannot guarantee neither uniqueness nor global solvability.

Theorem ($\mathcal{C}^{r-1}$ dependence)

Let $r \geq 2$ be arbitrarily fixed, $I \subset \mathbb{R}$ be a nonvoid interval, $\xi \in I$ be fixed, $v\colon I \times M \to \mathbb{R}^n$ be function which fulfills condition $(\mathcal{L})$ and assume that $v \in \mathcal{C}^r(I \times M)$. Denote $\varphi(t,x)$ the uniquely determined solution of the Cauchy problem

$$\begin{cases} \dot{x}(t) &= v(t,x) \\ x(\xi) &= x \end{cases}$$

for all $x \in M$, then for all fixed $t \in I$ the function

$$x \longmapsto \varphi(t,x)$$

is $r-1$ times continuously differentiable on M.

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Theorem

Under the assumptions of the previous theorem, $\varphi \in \mathcal{C}^{r-1}(I \times M)$.

As the following results shows, an even stronger regularity statement can be proved, c.f. Arnol'd [1, Section 32.6].

Theorem ($\mathcal{C}^r$ dependence)

Under the assumptions of Theorem 8, with the exception $r \geq 1$ we have the following (even stronger) implication

$$v \in \mathcal{C}^r \quad \Rightarrow \quad \varphi \in \mathcal{C}^r.$$

Proposition ($\mathcal{C}^1 \Rightarrow$ locally Lipschitz)

Let $M \subset \mathbb{R}^n$ be a domain and $f\colon M \to \mathbb{R}^n$ be a continuously differentiable mapping. Then for all compact, convex set $C \subset M$ the restriction of f to the set C is Lipschitz and the Lipschitz constant is

$$L = \sup_{x \in C} \|f_{*x}\|,$$

moreover, the supremum is attained on C, therefore, in fact max *can be written instead of* sup.

Theorem (Brouwer)

Let $M \subset \mathbb{R}^n$ be a nonempty, open set and $f\colon M \to \mathbb{R}^n$ be an injective and continuous mapping. Then the set $f(M) \subset \mathbb{R}^n$ is open, further, f is a homeomorphism from M onto $f(M)$.

Definition

A rectification of a direction field is a diffeomorphism mapping it into a field of parallel directions.
A field is said to be rectifiable if there exists a rectification for it.

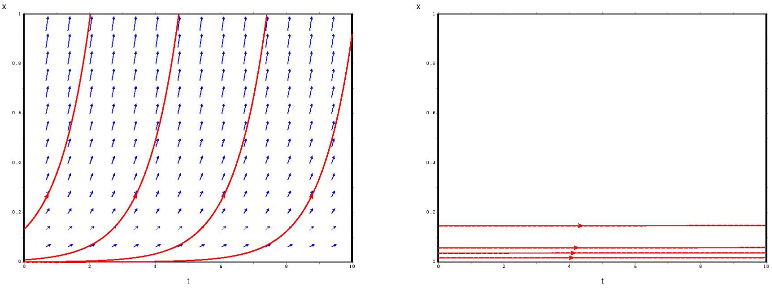
Example

The direction field of the equation

$$\dot{x} = x$$

is (globally) rectifiable via the diffeomorphism

$$\Phi(t, x) = \left(t, e^{-t} x\right)$$


x
t
0
0.2
0.4
0.6
0.8
1
2
4
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8
10

Theorem (Global rectification)

Let $I \subset \mathbb{R}$ be a nonempty open interval and $M \subset \mathbb{R}^n$ be an open set, $v \in \mathcal{C}^1(I \times M)$ be a function for which condition $(\mathcal{L})$ is fulfilled. Then the direction field of the equation

$$\dot{x} = v(t, x)$$

is globally rectifiable.

Fix $t_0 \in I$. Further, let $x \in M$ arbitrarily fixed and let us consider the following Cauchy problem

$$\begin{aligned} \dot{x}(t) &= v(t, x(t)) \\ x(t_0) &= x. \end{aligned}$$

Fix $t_0 \in I$. Further, let $x \in M$ arbitrarily fixed and let us consider the following Cauchy problem

$$\begin{aligned} \dot{x}(t) &= v(t, x(t)) \\ x(t_0) &= x. \end{aligned}$$

Due to the theorem of Picard and Lindelöf, there exists a uniquely determined solution to this problem, say, $\varphi(t, x)$. Furthermore, the mapping

$$M \ni x \longmapsto \varphi(t, x)$$

does exists and it depends on the variable x continuously differentiable. Additionally, due to Theorem 10, we also have $\varphi \in \mathcal{C}^1(I \times M)$.

Now, let us consider the mapping $\Phi\colon I \times M \to I \times \mathbb{R}^n$ defined by

$$\Phi(t,x) = (t, \varphi(t,x)) \qquad ((t,x) \in I \times M) .$$

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We will show that Φ is the desired rectification of the direction field. Indeed, we have

(i) The mapping Φ is differentiable, or even belongs to the class $\mathcal{C}^r(I \times M)$, if $v \in \mathcal{C}^r(I \times M)$.

(ii) Φ_{*x} maps the standard vector field, i.e., $1\dfrac{\partial}{\partial t} + 0\dfrac{\partial}{\partial x}$ into the field $1\dfrac{\partial}{\partial t} + v\dfrac{\partial}{\partial x}$.

(iii) Φ is a diffeomorphism from $I \times M$ onto $\Phi(I \times M)$.

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Proposition (ii) is straightforward.

Again, due to Theorem 10, we know that $\Phi \in \mathcal{C}^1(I \times M)$. Further, $I \times M \subset \mathbb{R}^{n+1}$ is an open set, Φ maps it onto $\Phi(I \times M) \subset \mathbb{R}^{n+1}$ continuously and we also know that (due to the theorem of Picard and Lindelöf) that Φ is injective. Thus, in view of the theorem of Brouwer, Φ is a differentiable homeomorphism. Since, Φ_{*x} is non-vanishing, Φ is a diffeomorphism.

In case the continuous function v does not fulfill condition $(\mathcal{L})$, we can only expect local and in general nonunique solvability. This might suggest that in such a case only the local version of the rectification theorem is valid.

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Corollary (Local rectification)

Every smooth direction field is rectifiable in a neighborhood of each point. If the field is r times continuously differentiable, then the rectifying diffeomorphism can also be taken from the class $\mathcal{C}^r$.

Let us assume that the differential equation corresponding to the direction field is

$$\dot{x} = v(t, x) \qquad (t, \in I, x \in M),$$

where $I \subset \mathbb{R}$ is a nonempty open interval, $M \subset \mathbb{R}^n$ is an open set and $v \in \mathcal{C}^1(I \times M)$.

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where $I \subset \mathbb{R}$ is a nonempty open interval, $M \subset \mathbb{R}^n$ is an open set and $v \in \mathcal{C}^1(I \times M)$.

Let $(t_0, x_0) \in I \times M$ be arbitrarily fixed. Then due to Theorem 2, there exists a compact and convex subset $J \times C \subset I \times M$ such that $v|_{J \times C}$ is a Lipschitz function, with Lipschitz constant

$$L = \sup_{\xi \in J \times C} \|v_{*x}\|\,.$$

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$$L = \sup_{\xi \in J \times C} \|v_{*x}\|\,.$$

Then clearly,

$$L \geq \sup_{\xi \in J^\circ \times C^\circ} \|v_{*x}\|\,,$$

where S° denotes the interior of the set S.

This shows that the restriction of v to the open set $J^{\circ} \times C^{\circ}$ is a Lipschitz function and v is continuous on this set, too. Thus, the assumptions of the Global rectification theorem are fulfilled. Therefore that part of direction field which belongs to $J^{\circ} \times C^{\circ}$ is rectifiable.

This shows that the restriction of v to the open set $J^\circ \times C^\circ$ is a Lipschitz function and v is continuous on this set, too. Thus, the assumptions of the Global rectification theorem are fulfilled. Therefore that part of direction field which belongs to $J^\circ \times C^\circ$ is rectifiable. Since $(t_0, x_0) \in I \times M$ was arbitrary, we get that every point of the extended direction field admits a neighborhood in which the direction field is rectifiable.

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