Finsler 2-manifolds with maximal holonomy group of infinite dimension

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joint work with P.T. Nagy

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Parallel translation, holonomy

- M is simply connected
- Finslerian metric: $g = g_{ij}(\mathbf{x},\mathbf{y})dx^i \otimes dx^j$
- Geodesics: $\ddot{x}^i + 2G^i(x,\dot{x}) = 0, \qquad G^i := \frac{1}{4}g^{il}\Big(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\Big)y^j y^k.$
- Parallel vector field $X(t)$ along a curve $c(t)$:

$$\nabla_{\dot{c}}X(t) = \Big(\frac{dX^i(t)}{dt} + \Gamma^i_j(c(t),X(t))\dot{c}^j(t)\Big)\frac{\partial}{\partial x^i} = 0, \quad \Gamma^i_j = \frac{\partial G^i}{\partial y^j}.$$

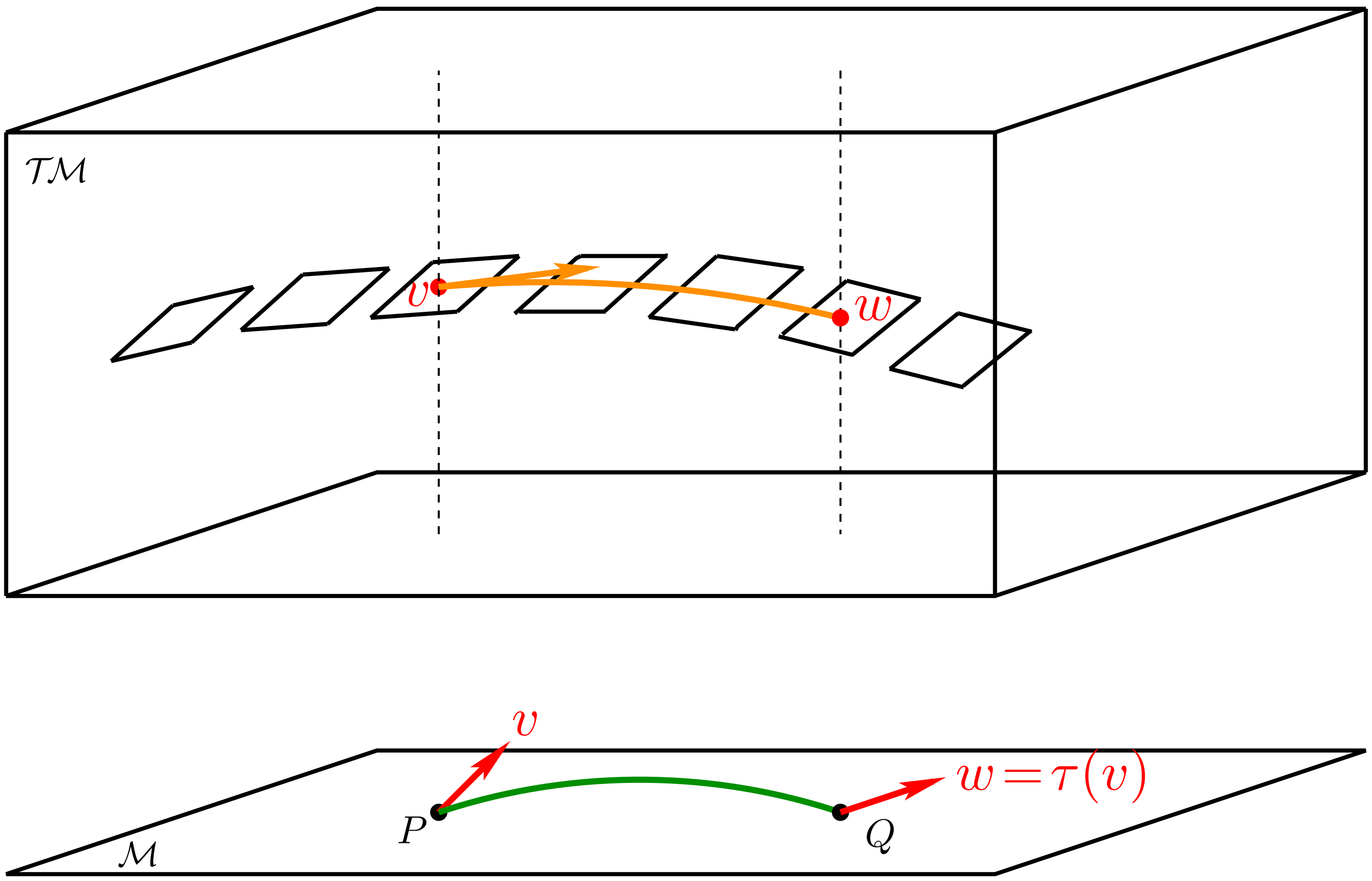
- Parallel translation along a curve $c\colon [0,1] \to M$:

$$\tau_c\colon T_{c_0}M \to T_{c_1}M, \quad \Rightarrow \begin{cases} \tau(\lambda v) = \lambda\tau(v) \\ \|\tau(v)\| = \|v\| \end{cases} \Rightarrow \tau_c : \mathcal{I}_{c_0} \to \mathcal{I}_{c_1}$$

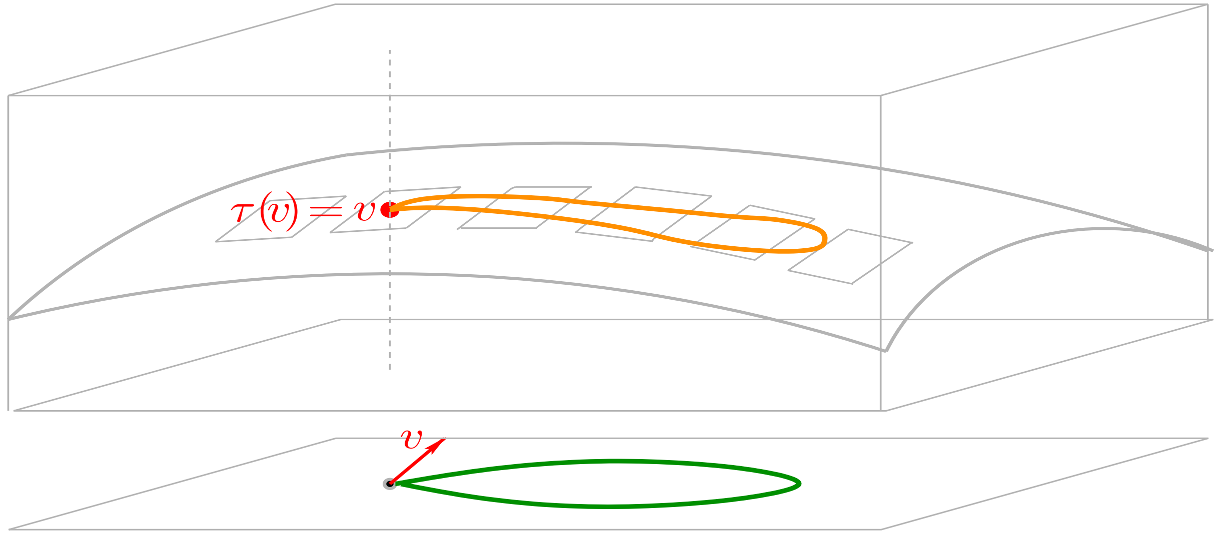
- The *holonomy group* is generated by parallel translation along closed curves

$\Rightarrow$ subgroup of $\mathrm{Diff}^\infty_+(\mathcal{I}_x)$ determined by parallel translations.

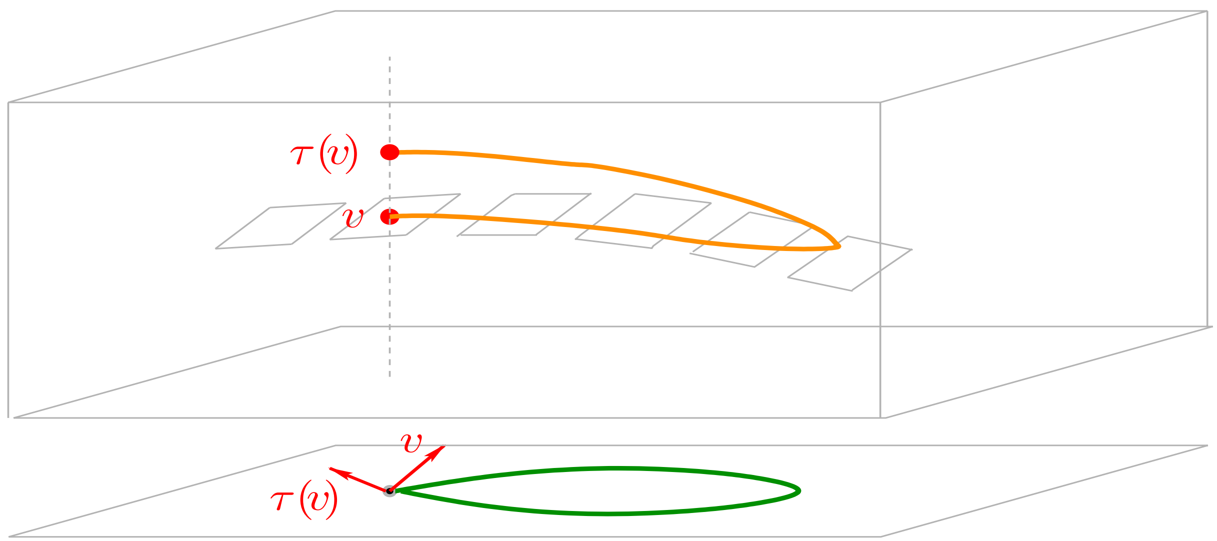
Parallel translation: geometric construction



- $R \equiv 0$



- $R \not\equiv 0$



Tangent Lie algebras to a subgroup H of $\mathrm{Diff}^\infty(\mathcal{I})$

Def: • A vector field X is ***tangent*** to H, if there exists a differentiable curve of diffeomorphisms $\{\phi_t\}$ in H such that

$$\phi_0 = \mathrm{Id}, \qquad \left.\frac{\partial \phi_t}{\partial t}\right|_{t=0} = X.$$

• A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{X}^\infty(\mathcal{I})$ is called ***tangent*** to H, if all elements of $\mathfrak{h}$ are tangent to H.

$$\boxed{\mathfrak{h} \text{ tangent to } H \quad \Rightarrow \quad \text{information on } H}$$

Property: If $\mathfrak{h}$ is tangent to a closed subgroup H, then

$$\boxed{\exp(\mathfrak{h}) \subset H}$$

Definition: A vector field X is ***strongly tangent*** to a subgroup H, if there exists a $k \in \mathbb{N}$ and a smooth k-parameter family $\{\phi_{(t_1,\ldots,t_k)}\}$ of diffeomorphisms in H such that

1. $\phi_{(t_1,\ldots,t_k)} = \mathrm{Id}$, if $t_j = 0$ for some $1 \leq j \leq k$;
2. $\displaystyle \frac{\partial^k \phi_{(t_1,\ldots,t_k)}}{\partial t_1 \ldots \partial t_k}\Big|_{(t_1,\ldots,t_k)=(0,\ldots,0)} = X.$

Proposition: *The Lie algebra generated by strongly tangent vector fields is tangent to H.*

$$\begin{array}{l} X_1 \text{ strongly tangent} \Rightarrow \{\phi^1_{(t_1,\ldots,t_{k_1})}\} \\ X_2 \text{ strongly tangent} \Rightarrow \{\phi^2_{(t_1,\ldots,t_{k_2})}\} \end{array} \quad \Rightarrow \quad \left[\phi^1_{(t_1,\ldots,t_{k_1})}, \phi^2_{(t_1,\ldots,t_{k_2})}\right]$$

$$\left[\phi^1_{(t_1,\ldots,t_{k_1})}, \phi^2_{(t_1,\ldots,t_{k_2})}\right] = \left(\phi^1_{(t_1,\ldots,t_{k_1})}\right)^{-1} \circ \left(\phi^2_{(t_1,\ldots,t_{k_2})}\right)^{-1} \circ \left(\phi^1_{(t_1,\ldots,t_{k_1})}\right) \circ \left(\phi^2_{(t_1,\ldots,t_{k_2})}\right)$$

$$\left[\phi^1_{(t_1,\ldots,t_{k_1})}, \phi^2_{(t_1,\ldots,t_{k_2})}\right] \quad \Rightarrow \quad [X_1, X_2]$$

Tangent Lie algebras to the $\mathrm{Hol}_x(M)$

Proposition: $\mathfrak{R}_x(M)$ and $\mathfrak{hol}^*_x(M)$ are tangent to $\mathrm{Hol}_x(M)$.

- $\mathfrak{R}(M)$: the *curvature algebra* is the smallest Lie algebra generated by curvature vector fields.

- $\mathfrak{hol}^*(M)$: the *infinitesimal holonomy algebra* is the smallest Lie algebra generated by curvature vector fields and by horizontal Berwald differentiation.

Projectively flat Finsler surfaces of constant curvature

Remarks:

- $\dim M = 2$,
- $\mathcal{I}_x \simeq \mathbb{S}^1$,
- $\mathsf{Hol}_x(M) \subset \mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$
- $\dim \mathfrak{R}_x(M) \leq 1$,
- $\mathfrak{hol}^*_x(M)$ can be higher (even infinite) dimensional,
- $G^i = \mathcal{P}(x,y)y^i$,
- $R^i_{jk} = \lambda\big(\delta^i_j g_{km} y^m - \delta^i_k g_{jm} y^m\big)$.

Theorem. The holonomy group of a projectively flat, spherically symmetric Finsler 2-manifolds of constant curvature is maximal:

$$\overline{\mathsf{Hol}(M)} = \mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1).$$

Proof :there exists $x_\mathrm{o} \in M$, where $\mathcal{F}(x_\mathrm{o}, y) = \|y\|$ and $\mathcal{P}(x_\mathrm{o}, y) = c\cdot\|y\|$

- $\mathcal{I}_{x_\mathrm{o}} = \mathbb{S}^1$, $\quad \mathsf{Hol}_\mathrm{o}(M) \subset \mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$
- $\mathfrak{hol}^*_{x_\mathrm{o}}(M) \supset \mathsf{F}(\mathbb{S}^1) = \left\{\cos nt\,\frac{\partial}{\partial t}, \sin nt\,\frac{\partial}{\partial t}\right\}_{n\in\mathbb{N}} \Rightarrow \overline{\mathsf{F}(\mathbb{S}^1)} = \overline{\mathfrak{hol}^*_{x_\mathrm{o}}(M)} = \mathfrak{X}(\mathbb{S}^1)$
- $\exp(\mathfrak{X}(\mathbb{S}^1)) = \exp\left(\overline{\mathfrak{hol}^*_\mathrm{o}(M)}\right) \subset \overline{\exp(\mathfrak{hol}^*_\mathrm{o}(M))} \subset \mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$
- $\left\langle\exp(\mathfrak{X}(\mathbb{S}^1))\right\rangle \subset \left\langle\,\overline{\exp(\mathfrak{hol}^*_\mathrm{o}(M))}\,\right\rangle \subset \overline{\mathsf{Hol}_\mathrm{o}(M)} \subset \mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$
- $\left\langle\exp(\mathfrak{X}(\mathbb{S}^1))\right\rangle$ conj. inv. $\Rightarrow$ normal subgroup in $\mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$
- $\mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$ simple

$\Rightarrow \left\langle\exp(\mathfrak{X}(\mathbb{S}^1))\right\rangle = \mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1) \Rightarrow \overline{\mathsf{Hol}_\mathrm{o}(M)} = \mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$

Corollary: The holonomy group of the Funk metric (constant negative curvature) and of the Bryant-Shen 2-spheres (constant positive curvature) are maximal.

Projectively flat Finsler surfaces of constant curvature

Theorem: The holonomy group of a locally projectively flat Finsler surface of constant curvature is finite dimensional if and only if

1. $R = 0$,
2. $R \neq 0$ and the associated canonical connection is linear.

- If $\lambda \neq 0$, ∇ is nonlinear: suppose that Hol(M) is finite dimensional:

  S. Lie: If a finite-dimensional connected Lie group acts on a 1-dimensional manifold without fixed points, than its dimension is less than 4.

  · $x_0 \in M$, $\xi = R_{x_0}(X, Y)$

  · ∇ nonlinear $\Rightarrow \{\xi,\ \nabla_1\xi,\ \nabla_2\xi\}$ $\mathbb{R}$-linearly independent,

  · $\{\xi,\ \nabla_1\xi,\ \nabla_2\xi,\ \nabla_i\nabla_j\xi\}$ $\mathbb{R}$-linearly dependent,

  · $\left\{1,\ \frac{\partial\mathcal{P}}{\partial y^1},\ \frac{\partial\mathcal{P}}{\partial y^2},\ 2\frac{\partial\mathcal{P}}{\partial y^i}\frac{\partial\mathcal{P}}{\partial y^j} - \lambda g_{ij}\right\}$ $\mathbb{R}$-linearly dependent

  · $\lambda g_{ij} = 2\mathcal{P}_i\mathcal{P}_j + A_{ij} + B_{ij}^m\mathcal{P}_m, \qquad A_{ij}, B_{ij} \in \mathbb{R}$,

  · $g_{ij} = \partial_{y^i y^j}E \quad \Rightarrow \quad \partial_i g_{jk} - \partial_k g_{ij} = 0 \quad \Rightarrow$ PDE on $\mathcal{P}$

$$\begin{cases} 2\mathcal{P}_2\mathcal{P}_{11} - 2\mathcal{P}_1\mathcal{P}_{12} + b_2\mathcal{P}_{11} + (c_2 - b_1)\mathcal{P}_{12} - c_1\mathcal{P}_{22} = 0, \\ 2\mathcal{P}_1\mathcal{P}_{22} - 2\mathcal{P}_2\mathcal{P}_{12} - b_3\mathcal{P}_{11} + (b_2 - c_3)\mathcal{P}_{12} + c_2\mathcal{P}_{22} = 0. \end{cases}$$

$$\mathcal{P}(x_0, y) = y_2 \cdot f\left(\frac{y_1}{y_2}\right) \quad \Rightarrow \quad \begin{cases} f''\left(\frac{2}{y_2}f + \frac{b_2}{y_2} + (b_1 - c_2)\frac{y_1}{y_2^2} + \frac{c_1 y_1^2}{y_2^3}\right) = 0, \\ f''\left(\frac{2y_1}{y_2^2}f - \frac{b_3}{y_2} + (c_3 - b_2)\frac{y_1}{y_2^2} + \frac{c_2 y_1^2}{y_2^3}\right) = 0. \end{cases}$$

$$t = \frac{y_1}{y_2} \qquad \Rightarrow \qquad \begin{cases} 2f + b_2 + (b_1 - c_2)t + c_1 t^2 = 0, \\ 2tf - b_3 + (c_3 - b_2)t + c_2 t^2 = 0. \end{cases}$$

$$b_3 + (2b_2 - c_3)t - (2c_2 - b_1)t^2 + c_1 t^3 \equiv 0,$$

$$f(t) = -b_2 - c_2 t, \qquad \mathcal{P}(x_0, y) = -y_2 b_2 - c_2 y_1$$

$\Rightarrow$ $\mathcal{P}$ is linear $\Rightarrow$ ∇ is linear $\Rightarrow$ contradiction.

$\Rightarrow$ $\mathrm{Hol}_{x_o}(M)$ is infinite dimensional.

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