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Some numerical characteristics of Sylvester and Hadamard matrices

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Abstract. We introduce numerical characteristics of *Sylvester* and *Hadamard matrices* and give their estimates and some of their applications.

1. Introduction

A Hadamard matrix has a simple structure, it is a square matrix such that its entries are either $+1$ or -1 and its rows (columns) are mutually orthogonal. In spite of the fact that Hadamard matrices have been actively studied for about 150 years, they still have unknown properties. If $\mathcal{H}_n$ is a Hadamard matrix of order n, then the matrix

$$\begin{bmatrix} \mathcal{H}_n & \mathcal{H}_n \\ \mathcal{H}_n & -\mathcal{H}_n \end{bmatrix}$$

is a Hadamard matrix of order $2n$. Applying this algorithm repeatedly J. J. Sylvester has constructed a particular sequence of Hadamard matrices of order 2^n. These matrices are called Sylvester matrices or Walsh matrices.

Hadamard matrices have a wide range of applications in the code theory, scheduling theory, statistics, modern communications etc. In this paper we deal with its application for certain problems of functional analysis. Namely, using

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Hadamard matrices, in the classical Banach spaces it is easy to construct examples of unconditionally convergent series which do not converge absolutely (see [1], [12], [13]). Note that in [8] the author made a considerable effort to prove the existence of such series in the space l_1 without giving construction. To prove the unconditional convergence of the above mentioned constructed series the numerical characteristics of Hadamard and Sylvester matrices are important tools. In the present paper the general forms of these tools for Banach spaces with bases are considered. These characteristics and the structure of the Hadamard and Sylvester matrices play an important role in the investigation of the convergence of series in Banach spaces (see e.g., [2], [14], [15]). For these characteristics we give estimates (cf. Theorems 3.1, 3.6, 4.2 and 4.8). We believe that the investigated characteristics and their estimates complete our knowledge about Hadamard matrices and may have applications in other fields of mathematics.

In Section 2 some concepts, definitions and auxiliary results required for further discussions are given.

In Section 3 the numerical characteristic $\varrho^{(n)}$ of Sylvester matrices is introduced and its estimates for the case of a Banach space with a subsymmetric basis (φ_i) are studied. For every positive integer n we prove the following estimates (cf. Theorems 3.1 and 3.6)

$$\max\left\{\frac{n+2}{6}\cdot\lambda(2^n),2^n\right\}\le\varrho^{(n)}\le\min\left\{\left(1+\sum_{j=1}^{n}2^{-j}\lambda(2^{j-1})\right)\cdot 2^n,\lambda(n)\cdot 2^n\right\},$$

where $\lambda(n)=\|\sum_{i=1}^{n}\varphi_i\|$.

In Section 4 we define the analogue characteristic ϱ_n for Hadamard matrices. For every positive integer n for which there is a Hadamard matrix of order n we show the following estimates (cf. Theorems 4.2 and 4.8)

$$\max\left\{(1/\sqrt{2}\,)\lambda(n)\sqrt{n},n\right\}\le\varrho_n\le\lambda([\sqrt{n}\,]+1)n,$$

where $[\sqrt{n}\,]$ is the integer part of $\sqrt{n}$.

As an application of the introduced notions we give a characterization for the spaces isomorphic to l_1 in terms of these characteristics (cf. Theorem 4.10).

In Section 5 we pose an open problem which has naturally arisen from our investigations.

Most of the results of this paper were announced in [5] without proofs. Here these results and some new ones are given with complete proofs.

2. Notation and preliminaries

We follow the standard notation and terminology used, for example, in [7]. The notations c_0, l_p and $L_p, 1 \leq p < \infty$, have their usual meaning.

A sequence (φ_i) of nonzero elements in a real Banach space X is called a (*Schauder*) *basis* of X if for every $x \in X$ there is a unique sequence of scalars (α_i) so that $x = \sum_{i=1}^{\infty} \alpha_i \varphi_i$. If (φ_i) is a basis in a Banach space X with a norm $\|\cdot\|$, then there is a constant $K \geq 1$ so that for every choice of scalars (α_i) and positive integers $n < m$, we have

$$\Big\| \sum_{i=1}^{n} \alpha_i \varphi_i \Big\| \leq K \Big\| \sum_{i=1}^{m} \alpha_i \varphi_i \Big\|.$$

The smallest possible constant K in this inequality is called the *basis constant* of (φ_i). Note that in X there exists an *equivalent norm* $|||\cdot|||$ (i.e. for some positive constants C_1, C_2: $C_1\|x\| \leq |||x||| \leq C_2\|x\|$ for every $x \in X$) for which the basis constant is $K = 1$.

A basis (φ_i) is called *normalized* if $\|\varphi_i\| = 1$ for all i. Let (φ_i) be a basis of a Banach space X. A sequence of linear bounded functionals (φ_i^*) defined by the relation $\langle \varphi_i^*, \varphi_j \rangle = \delta_{ij}$, where δ_{ij} is the *Kronecker delta*, is called the sequence of *biorthogonal functionals* associated to the basis (φ_i). Two bases, (φ_i) of X and (ψ_i) of Y, are called equivalent provided a series $\sum_{i=1}^{\infty} \alpha_i \varphi_i$ converges if and only if $\sum_{i=1}^{\infty} \alpha_i \psi_i$ converges.

A basis (φ_i) of a Banach space X is *unconditional* if for any permutation $\pi : \mathbb{N} \to \mathbb{N}$ of the set $\mathbb{N}$ of positive integers $(\varphi_{\pi(i)})$ is a basis of X. If (φ_i) is an unconditional basis of a real Banach space X, then there is a constant $K \geq 1$ so that for every choice of scalars (α_i) for which $\sum_{i=1}^{\infty} \alpha_i \varphi_i$ converges and every choice of bounded scalars (λ_i) we have

$$\Big\| \sum_{i=1}^{\infty} \lambda_i \alpha_i \varphi_i \Big\| \leq K \sup_i |\lambda_i| \Big\| \sum_{i=1}^{\infty} \alpha_i \varphi_i \Big\|.$$

The smallest possible constant K in this inequality is called the *unconditional constant* of (φ_i). If (φ_i) is an unconditional basis of X, then there is an equivalent norm in X so that the unconditional constant becomes 1.

The sequence of unit vectors $e_i = (0, 0, \ldots, \overset{i}{1}, 0, \ldots)$, $i = 1, 2, \ldots$, is an example of an unconditional basis in c_0 and l_p, $1 \leq p < \infty$ (the basis (e_i) is called the *natural basis* of the corresponding spaces). The *Haar system* is an unconditional basis in the function spaces $L_p(0,1)$, $1 < p < \infty$. This system is also basis in $L_1(0,1)$, but in this space there does not exist an unconditional basis.

Every normalized unconditional basis in l_1, l_2 or c_0 is equivalent to the natural basis of these spaces. Moreover, a Banach space has, up to equivalence, a unique unconditional basis if and only if it is isomorphic to one of the following three spaces: l_1, l_2 or c_0.

Let $(X, \|\cdot\|)$ be a Banach space with a normalized basis (φ_i). Consider the expression

$$\lambda(n) = \Big\| \sum_{i=1}^{n} \varphi_i \Big\|, \quad n = 1, 2, \dots .$$

For every space having an unconditional basis whose unconditional constant is 1 with the exception of the space c_0 we have that $(\lambda(n))$ is a non-decreasing sequence and $\lim_{n\to\infty} \lambda(n) = \infty$. More precisely, if $\sup_n \lambda(n) < \infty$, then (φ_i) is equivalent to the natural basis of the space c_0 (see, for example, [7], p. 120).

A basis (φ_i) of a Banach space X is said to be *symmetric* if for any permutation π of the positive integers $(\varphi_{\pi(i)})$ is equivalent to (φ_i). If (φ_i) is a symmetric basis of a Banach space X, then there is a constant K such that for any choice of scalars (α_i) for which $\sum_{i=1}^{\infty} \alpha_i \varphi_i$ converges, every choice of signs $\vartheta = (\vartheta_i)$ and any permutation π of the positive integers we have

$$\Big\| \sum_{i=1}^{\infty} \vartheta_i \alpha_i \varphi_{\pi(i)} \Big\| \le K \Big\| \sum_{i=1}^{\infty} \alpha_i \varphi_i \Big\|.$$

The smallest possible constant K in this inequality is called the *symmetric constant* of (φ_i).

A basis (φ_i) of a Banach space X is called *subsymmetric* if it is unconditional and for every increasing sequence of integers (i_n), (φ_{i_n}) is equivalent to (φ_i). If (φ_i) is a subsymmetric basis of a Banach space X, then there is a constant K such that for any choice of scalars (α_i) for which $\sum_{i=1}^{\infty} \alpha_i \varphi_i$ converges, every choice of signs $\vartheta = (\vartheta_i)$ and every increasing sequence of integers (i_n) we have

$$\Big\| \sum_{n=1}^{\infty} \vartheta_n \alpha_n \varphi_{i_n} \Big\| \le K \Big\| \sum_{i=1}^{\infty} \alpha_i \varphi_i \Big\|.$$

The smallest possible constant K in this inequality is called the *subsymmetric constant* of (φ_i).

Every symmetric basis is subsymmetric. The converse of this assertion is not true. The unit vectors in l_p, $1 \le p < \infty$, and c_0 are examples of symmetric basis.

Proposition 2.1 (see [7], Proposition 3.a.7, p. 119)**.** *Let $(X, \|\cdot\|)$ be a Banach space with a symmetric basis (φ_i) whose symmetric constant is equal to 1. Then there exists a new norm $\|\cdot\|_0$ on X such that:*

(a). $\|x\| \le \|x\|_0 \le 2\|x\|$ *for all* $x \in X$;

(a). *The symmetric constant of* (φ_i) *with respect to* $\|\cdot\|_0$ *is equal to 1;*

(c). *If we put* $\lambda_0(n) = \|\sum_{i=1}^n \varphi_i\|_0$, $n = 1, 2, \ldots$, *then* $\{\lambda_0(n+1) - \lambda_0(n)\}$ *is a non-increasing sequence, i.e.* $\lambda_0(\cdot)$ *is a concave function on the integers.*

The converse of the last assertion is also true in the sense that, for every concave non-decreasing sequence of positive numbers (λ_k) there exists at least one Banach space X having a symmetric basis (φ_i) with symmetric constant equal to 1 such that $\|\sum_{i=1}^n \varphi_i\| = \lambda_n$ for every n.

Proposition 2.2 (see [7], Proposition 3.a.4, p. 116)**.** (**A**). *Let* X *be a Banach space with a normalized subsymmetric basis* (φ_i) *whose subsymmetric constant is 1. Then the following inequality is valid*

$$\Big\| \sum_{i=1}^n \alpha_i \varphi_i \Big\| \ge \frac{\sum_{i=1}^n |\alpha_i|}{n} \lambda(n), \quad n = 1, 2, \ldots .$$

(**B**). *Moreover, if* (φ_i) *is a subsymmetric basis, then one has*

$$\Big\| \sum_{i=1}^n \alpha_i \varphi_i \Big\| \ge \frac{\sum_{i=1}^n |\alpha_i|}{2n} \lambda(n), \quad n = 1, 2, \ldots .$$

From this it follows that if $\lim_{n\to\infty} \sup \lambda(n)/n > 0$, then (φ_i) is equivalent to the natural basis of the space l_1 (see, for example, [7], p. 20).

The *Rademacher functions* r_k, $k = 1, 2, \ldots$, are defined on $[0, 1]$ by the equality

$$r_k(t) = sign(\sin 2^k \pi t).$$

Let us note the well-known *Khintchine's inequality*: for every $0 < p < \infty$ there exist positive constants A_p and B_p so that

$$A_p \Big(\sum_{k=1}^m |\alpha_k|^2 \Big)^{1/2} \le \Big(\int_0^1 \Big| \sum_{k=1}^m \alpha_k r_k(t) \Big|^p dt \Big)^{1/p} \le B_p \Big(\sum_{k=1}^m |\alpha_k|^2 \Big)^{1/2},$$

$m = 1, 2, \ldots$, for every choice of scalars $(\alpha_1, \alpha_2, \ldots, \alpha_m)$. For $p = 1$ the best constant is $A_1 = 1/\sqrt{2}$ (see [10]).

A Banach space X is said to be of *type p* if there is a constant $T_p = T_p(X) \ge 0$ such that for any finite collection of vectors $x_1, x_2, \ldots, x_n$ in X we have

$$\Big(\int_0^1 \Big\| \sum_{k=1}^n r_k(t) x_k \Big\|^2 dt \Big)^{1/2} \le T_p \Big(\sum_{k=1}^n \|x_k\|^p \Big)^{1/p}, \quad n = 1, 2, \ldots .$$

In the Khintchine's inequality the notion of type p has meaning for the case $0 < p \leq 2$. Every Banach space has type p for $0 < p \leq 1$. The spaces l_p, $L_p([0,1])$, $1 \leq p < \infty$, have type $\min(2,p)$.

A Hadamard matrix is a square matrix of order n with entries ± 1 such that any two columns (rows) are orthogonal (see *e.g.* [4], p. 238, [9], p. 44). We denote by $\mathcal{H}_n = [h^n_{ki}]$ a Hadamard matrix of order n. It is easy to see that the order of a Hadamard matrix is either 1 or 2 or it is divisible by 4. *Hadamard* put forward the conjecture that for any n divisible by 4 there exists a Hadamard matrix of order n. As far as we know, *Hadamard's conjecture* remains open. Let $\mathbb{N}_{\mathcal{H}}$ be the set of all positive integers n for which there exists a Hadamard matrix of order n.

The following property follows from the definition of Hadamard matrices. If $\mathcal{H}_n = [h^n_{ki}]$ is a Hadamard matrix, then for every n, $n \in \mathbb{N}_{\mathcal{H}}$, we have

$$\sum_{i=1}^{n} h^n_{ki} h^n_{mi} = n\, \delta_{km}, \quad \sum_{k=1}^{n} h^n_{ki} h^n_{kj} = n\, \delta_{ij}.$$

Therefore for any n, $n \in \mathbb{N}_{\mathcal{H}}$, and every sequence $(\beta_i)_{i \leq n}$ of real numbers one has

$$\sum_{k=1}^{n} \left(\sum_{i=1}^{n} h^n_{ki} \beta_i \right)^2 = n \sum_{i=1}^{n} \beta_i^2.$$

It is easy to see that multiplying any row or any column of a Hadamard matrix by -1 we get again a Hadamard matrix.

Let the triple $(\Omega, \mathfrak{A}, \mathbb{P})$ be a probability space, where Ω be a non-empty set, $\mathfrak{A}$ be a σ-algebra of subsets of Ω and $\mathbb{P}$ be a probability measure on the measurable space $(\Omega, \mathfrak{A})$, (i.e. $\mathbb{P}$ is assumed to be a non-negative measure on $(\Omega, \mathfrak{A})$ satisfying the condition $\mathbb{P}(\Omega) = 1$). Let X be a real Banach space with the topological dual space X^*. A function $\xi : \Omega \to X$ is scalarly measurable (respectively scalarly integrable) if for each $x^* \in X^*$ the scalar function $\langle x^*, \xi \rangle$ is measurable (respectively integrable, i.e. $\langle x^*, \xi \rangle \in L_1(\Omega, \mathfrak{A}, \mathbb{P})$). A scalarly integrable function $\xi : \Omega \to X$ is *Pettis integrable* (or *weak integrable*) if for each $A \in \mathfrak{A}$ there exists a vector $m_{\xi,A} \in X$ such that for every $x^* \in X^*$ we have

$$\langle x^*, m_{\xi,A} \rangle = \int_A \langle x^*, \xi \rangle \, d\mathbb{P}.$$

For a Pettis integrable function $\xi : \Omega \to X$ the element $m_{\xi,\Omega}$ is called the *Pettis integral* of ξ with respect to $\mathbb{P}$. It is also called the *mean value* of the function ξ. We denote by $\mathbb{E}\,\xi$ the Pettis integral of the function ξ. If a function $\xi : \Omega \to X$ has a measurable norm and there exists $\mathbb{E}\,\xi$, then $\|\mathbb{E}\,\xi\| \leq \mathbb{E}\,\|\xi\|$. For every separably valued function $\xi : \Omega \to X$ from the condition $\mathbb{E}\,\|\xi\| < \infty$ it follows the existence of the Pettis integral $\mathbb{E}\,\xi$ (ξ is separably valued if $\xi(\Omega)$ is a separable subset of X).

For details and proofs related with the topics of this section see [7] and [11].

3. Sylvester matrices

The *Sylvester matrices* are special cases of Hadamard matrices. They are defined by the recursion relations (cf. [9], p. 45):

$$\mathcal{S}^{(1)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathcal{S}^{(n)} = \begin{bmatrix} \mathcal{S}^{(n-1)} & \mathcal{S}^{(n-1)} \\ \mathcal{S}^{(n-1)} & -\mathcal{S}^{(n-1)} \end{bmatrix}, \quad n = 2, 3, \ldots .$$

$\mathcal{S}^{(n)}$ is a Hadamard matrix of order 2^n and hence $2^n \in \mathbb{N}_{\mathcal{H}}$ for all $n = 1, 2, \ldots$.

If the first column of a Hadamard matrix $\mathcal{H}_n = [h^n_{ki}]$ consists of only $+1$, then one has

$$\sum_{k=1}^{n} h^n_{ki} = \begin{cases} n, & \text{for } i = 1, \\ 0, & \text{for } i = 2, 3, \ldots, n. \end{cases}$$

In particular, if $\mathcal{S}^{(n)} = [s^{(n)}_{ki}]$ is the Sylvester matrix of order 2^n, $n = 1, 2, \ldots$, then we get

$$\sum_{k=1}^{2^n} s^{(n)}_{ki} = \begin{cases} 2^n, & \text{for } i = 1, \\ 0, & \text{for } i = 2, 3, \ldots, 2^n \end{cases}$$

and

$$\sum_{k=1}^{2^{n-1}} s^{(n)}_{ki} = \begin{cases} 2^{n-1}, & \text{for } i = 1 \text{ and } i = 2^{n-1} + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{S}^{(n)} = [s^{(n)}_{ki}]$ be the Sylvester matrix of order 2^n, $n = 1, 2, \ldots$, and X be a Banach space with a norm $\|\cdot\|$ and a normalized basis (φ_i). Consider the function

$$\varrho^{(n)}(m) = \left\| \sum_{i=1}^{2^n} \left(\sum_{k=1}^{m} s^{(n)}_{ki} \right) \varphi_i \right\|, \quad m = 1, 2, \ldots, 2^n. \tag{3.1}$$

One has $\varrho^{(n)}(1) = \lambda(2^n)$, $\varrho^{(n)}(2) = 2 \,\| \sum_{i=1}^{2^{n-1}} \varphi_{2i-1} \|$, $\varrho^{(n)}(2^n) = 2^n$, where $\lambda(2^n) = \| \sum_{i=1}^{2^n} \varphi_i \|$. The function $\varrho^{(n)}(m)$ obviously depends on X, the norm in X and the choice of basis (φ_i). In particular, for the case of the spaces l_p, $1 \le p < \infty$, with respect to the natural basis $\varrho^{(n)}(m)$ has the form $(\sum_{i=1}^{2^n} | \sum_{k=1}^{m} s^{(n)}_{ki} |^p)^{1/p}$.

We set

$$\varrho^{(n)} = \max_{1 \le m \le 2^n} \varrho^{(n)}(m). \tag{3.2}$$

The function $\varrho^{(n)}(m)$ can be expressed as follows. Let $a_k = \sum_{i=1}^{2^n} s^{(n)}_{ki} \varphi_i$, $k = 1, 2, \ldots, 2^n$. Then one has $\varrho^{(n)}(m) = \| \sum_{k=1}^{m} a_k \|$. If (φ_i) is an unconditional basis with unconditional constant equal to 1, then, obviously, $\|a_k\| = \lambda(2^n)$ for any $k = 1, 2, \ldots, 2^n$ and $\varrho^{(n)} \le \lambda(2^n)\, 2^n \le 2^{2n}$.

In $l_p,\ 1 \leq p < \infty$, it was proved in [12] that $\varrho^{(n)} \leq n\,2^n$.

The following theorem gives a similar estimate of $\varrho^{(n)}$ in the case of general Banach spaces with subsymmetric basis.

Theorem 3.1. *Let X be a Banach space with normalized subsymmetric basis whose subsymmetric constant is* 1. *Then for $\varrho^{(n)}$ defined by* (3.2) *one has the following estimate*

$$\varrho^{(n)} \leq \min\left\{\left(1+\sum_{j=1}^{n} 2^{-j}\lambda(2^{j-1})\right)\cdot 2^n,\quad \lambda(n)\cdot 2^n\right\},\quad n=1,2,\dots. \tag{3.3}$$

Proof. First we prove the inequality $\varrho^{(n)} \leq (1+\sum_{j=1}^{n} 2^{-j}\lambda(2^{j-1}))\cdot 2^n$ by induction. For $n=1$ it is true since the left hand side of (3.3) is equal to 2 and the right hand side is equal to 3. Let $n \geq 2$. Introduce the following notation

$$\alpha_i^{(n)}(m) = \sum_{k=1}^{m} s_{ki}^{(n)},\quad 1 \leq i, m \leq 2^n. \tag{3.4}$$

Therefore we get

$$\alpha_1^{(n)}(m) = m \tag{3.5}$$

and

$$\alpha_{2^{n-1}+1}^{(n)}(m) = \begin{cases} m, & \text{for} \quad 1 \leq m \leq 2^{n-1}, \\ 2^n - m, & \text{for} \quad 2^{n-1}+1 \leq m \leq 2^n. \end{cases} \tag{3.6}$$

Since $i \leq 2^n$ we can write that $i = \varepsilon_n 2^n + \varepsilon_{n-1}2^{n-1} + \dots + \varepsilon_1 2 + \varepsilon_0$, where $\varepsilon_j \in \{0,1\}$ for every j. Then by the definition and the properties of the Sylvester matrices we can prove by induction that for any i

$$\max_{1\leq m\leq 2^n}\left|\alpha_i^{(n)}(m)\right| = 2^{f(i)}, \tag{3.7}$$

where the function $f : \{1,2,\dots,n\} \to \{0,1,2,\dots,n\}$ is defined as follows: $f(1) = n$; $f(i) = 0$ if $\varepsilon_0 = 0$ (i.e. i is an even number) and if $\varepsilon_0 = 1$ (i.e. i is an odd number), then for $f(i)$ we have: $\varepsilon_{f(i)} = 1$ and $\varepsilon_j = 0$ for every $j = 1,2,\dots,f(i)-1$.

For $i=1$ and $i = 2^{n-1}+1$ the equality (3.7) is valid since from the relations (3.5) and (3.6) it follows that $\max_{1\leq m\leq 2^n}|\alpha_1^{(n)}(m)| = 2^n$ and $\max_{1\leq m\leq 2^n}|\alpha_{2^{n-1}+1}^{(n)}(m)| = 2^{n-1}$. To prove (3.7) for the rest indexes i we use the following equalities

$$\max_{1\leq m\leq 2^{n+1}}\left|\alpha_{2^n+i}^{(n+1)}(m)\right| = \max_{1\leq m\leq 2^{n+1}}\left|\alpha_i^{(n+1)}(m)\right| = \max_{1\leq m\leq 2^n}\left|\alpha_i^{(n)}(m)\right| \tag{3.8}$$

for any $i = 2, 3, \ldots, 2^n$, which is a consequence of the definition and the properties of the Sylvester matrices. Every positive integer i, $1 \le i \le 2^{n+1}$, has the unique representation given by

$$i = \begin{cases} \varepsilon_n 2^n + \cdots + \varepsilon_1 2 + \varepsilon_0, & \text{for } \ 1 \le i \le 2^n, \\ 2^n + \varepsilon_n 2^n + \cdots + \varepsilon_1 2 + \varepsilon_0, & \text{for } \ 2^n + 1 \le i \le 2^{n+1}. \end{cases} \tag{3.9}$$

If i is an even number, then in (3.9) we have $\varepsilon_0 = 0$ and by (3.7) and (3.8) we obtain $\max_{1\le m\le 2^{n+1}} |\alpha_i^{(n+1)}(m)| = 1$. If i is an odd number and, in addition, $i \ne 1$ and $i \ne 2^n + 1$, then we can rewrite (3.9) as follows:

$$i = \begin{cases} \varepsilon_n 2^n + \ldots + \varepsilon_{j_0+1} 2^{j_0+1} + 2^{j_0} + 1, & \text{for } \ 3 \le i \le 2^n, \\ 2^n + \varepsilon_n 2^n + \cdots + \varepsilon_{j_0+1} 2^{j_0+1} + 2^{j_0} + 1, & \text{for } \ 2^n + 3 \le i \le 2^{n+1}, \end{cases}$$

where $j_0 = 1, 2, \ldots, n-1$. Using again relations (3.7) and (3.8) we certainly have $\max_{1\le m\le 2^{n+1}} |\alpha_i^{(n+1)}(m)| = 2^{j_0}$.

Applying now a simple combinatorial calculation we get that the number of indexes i, $1 \le i \le 2^n$, for which $\max_{1\le m\le 2^n} |\alpha_i^{(n)}(m)| = 2^j$, is equal to 2^{n-j-1} for $j = 0, 1, 2, \ldots, n-1$, and the equality $\max_{1\le m\le 2^n} |\alpha_i^{(n)}(m)| = 2^n$ is satisfied only for $i = 1$.

As the subsymmetric constant of the basis (φ_i) is 1, using (3.7), we obtain for every $m = 1, 2, \ldots, 2^n$ the following relations:

$$\begin{aligned} \varrho^{(n)}(m) &= \Big\| \sum_{i=1}^{2^n} |\alpha_i^{(n)}(m)| \varphi_i \Big\| \le \Big\| \sum_{i=1}^{2^n} \max_{1\le m\le 2^n} |\alpha_i^{(n)}(m)| \varphi_i \Big\| \\ &= \Big\| 2^n \varphi_1 + \sum_{j=1}^{n} 2^{n-j} \sum_{i=2^{j-1}}^{2^j-1} \varphi_{\pi(i+1)} \Big\|, \end{aligned} \tag{3.10}$$

where π is a permutation of a sequence of the positive integers $\{2, 3, \ldots, 2^n\}$. Applying now the triangular inequality on the right hand side of (3.10) and using the fact that (φ_i) is a subsymmetric basis we get the required inequality.

Now we prove the inequality $\varrho^{(n)} \le \lambda(n)\, 2^n$. The number of the (not necessarily different) basis elements involved in the right hand side of the inequality (3.10) is equal or less than $n \cdot 2^n$ (more exactly, $(1 + n/2) \cdot 2^n$). Hence we get the following equality

$$2^n \varphi_1 + \sum_{j=1}^{n} 2^{n-j} \sum_{i=2^{j-1}}^{2^j-1} \varphi_{\pi(i+1)} = \sum_{k=1}^{2^n} \sum_{i=1}^{l_k} \varphi_{k_i}, \tag{3.11}$$

where $1 \le l_k \le n$ for any $k = 1, 2, \ldots, 2^n$, $\varphi_{k_i} \in \{\varphi_1, \varphi_2, \ldots, \varphi_{2^n}\}$, for any fixed k and for every $i \ne j$, $i, j = 1, 2, \ldots, l_k$, we have $\varphi_{k_i} \ne \varphi_{k_j}$ and for any fixed i but for different indexes k the elements φ_{k_i} can be the same. As the basis (φ_i) is subsymmetric with subsymmetric constant equal to 1 and (3.10) and (3.11) are valid we obtain

$$\begin{aligned}\varrho^{(n)}(m) \le \Big\| 2^n \varphi_1 + \sum_{j=1}^{n} 2^{n-j} \sum_{i=2^{j-1}}^{2^j-1} \varphi_{\pi(i+1)} \Big\| \\ = \Big\| \sum_{k=1}^{2^n} \sum_{i=1}^{l_k} \varphi_{k_i} \Big\| \le \sum_{k=1}^{2^n} \Big\| \sum_{i=1}^{l_k} \varphi_{k_i} \Big\| = \sum_{k=1}^{2^n} \lambda(l_k) \le \lambda(n)\, 2^n\end{aligned}$$

for every m. This proves the theorem. □

Remark 3.2. For the estimates proved in Theorem 3.1 with respect to the natural basis we obtain the relation $1 + \sum_{j=1}^{n} 2^{-j} \lambda(2^{j-1}) \le \lambda(n)$ in the case of $X = l_1$, but we have the converse relation $1 + \sum_{j=1}^{n} 2^{-j} \lambda(2^{j-1}) \ge \lambda(n)$ in the case of $X = c_0$.

Let X be a Banach space (not necessarily with basis), $x_1, x_2, \dots, x_{2^n}$ be a sequence of elements from the unit ball of X and $\mathcal{S}^{(n)}$ be the Sylvester matrix of order 2^n, $n = 1, 2, \dots$. By analogy with the definition of $\varrho^{(n)}$ let $\hat\varrho^{(n)}(m) = \| \sum_{i=1}^{2^n} (\sum_{k=1}^{m} s_{ki}^{(n)}) x_i \|$, $m = 1, 2, \dots, 2^n$, and let $\hat\varrho^{(n)} = \max_{1\le m\le 2^n} \hat\varrho^{(n)}(m)$.

Corollary 3.3. *We have* $\hat\varrho^{(n)} \le n \cdot 2^n$.

Proof. Using the triangular inequality and the fact that $\|x_i\| \le 1$ for any i, we have

$$\hat\varrho^{(n)}(m) \le \sum_{i=1}^{2^n} \Big| \sum_{k=1}^{m} s_{ki}^{(n)} \Big|.$$

The right hand side of the last relation is the expression $\varrho^{(n)}(m)$ in the space l_1 with respect to the natural basis, which is for every $m = 1, 2, \dots, 2^n$ less or equal than $n\, 2^n$ (cf. Theorem 3.1). □

Corollary 3.4. *Let X be a Banach space of type p, $p > 1$, with a normalized subsymmetric basis (φ_i) whose subsymmetric constant is 1. Then one has*

$$\varrho^{(n)} \le c \cdot 2^n,$$

where the constant $c \ge 1$ depends only on the space X.

Proof. Since (φ_i) is a normalized subsymmetric basis whose subsymmetric constant is 1, then $\lambda(2^{j-1}) \le T_p(X)\, 2^{(j-1)/p}$ for every $j \ge 1$, where $T_p(X)$ is the constant involved in the definition of the space of type p. Then for the right hand side of (3.3) we get

$$1+\sum_{j=1}^{n} 2^{-j}\lambda(2^{j-1}) \le 1+T_p(X)\sum_{j=1}^{n} 2^{-j+(j-1)/p} \le 1+T_p(X)/(2-2^{1/p}).$$

Taking $c = 1+T_p(X)/(2-2^{1/p})$ the proof is finished. □

Let us note that in the space c_0 we have a similar estimate, namely $\varrho^{(n)} \le 2^n$ (cf. Theorem 3.1), although c_0 is a space of type 1. As $\varrho^{(n)} \ge 2^n$, we get $\varrho^{(n)} = 2^n$ in the space c_0.

Thus, in the Banach spaces of type p, $p > 1$, (as well as in c_0), we have $\sup_n \varrho^{(n)}/2^n < \infty$. But in general this is not true. The following statement shows the validity of this fact for the space l_1.

Theorem 3.5 ([6]). *For the space l_1 with the natural basis one has*

$$\varrho^{(n)} = \max_{1\le m\le 2^n} \varrho^{(n)}(m) = (3n+7)2^n/9 + 2(-1)^n/9, \quad n \ge 1.$$

For any n the maximum is attained at the points $m_n = (2^{n+1}+(-1)^n)/3$ and $m_n' = (5\cdot 2^{n-1}+(-1)^{n-1})/3$.

Let us estimate $\varrho^{(n)}$ from below.

Theorem 3.6. *If a Banach space X satisfies the conditions of Theorem 3.1, then one has*

$$\varrho^{(n)} \ge \max\left\{\frac{n+2}{6}\,\lambda(2^n),\ 2^n\right\}, \quad n = 1,2,\dots .$$

Proof. By the definition of $\varrho^{(n)}(m)$ for any positive integer n we have, $\varrho^{(n)} \ge \varrho^{(n)}(2^n) = 2^n$ and the inequality $\varrho^{(n)} \ge 2^n$ is evident.

Let us prove that for any integer n the inequality $\varrho^{(n)} \ge \frac{n+2}{6}\,\lambda(2^n)$ is also true. Using the inequality of Proposition 2.2 (**B**) for any integer n we have

$$\left\|\sum_{i=1}^{2^n}\left|\alpha_i^{(n)}(m)\right|\varphi_i\right\| \ge \frac{\sum_{i=1}^{2^n}|\alpha_i^{(n)}(m)|}{2^{n+1}}\,\lambda(2^n) \quad \text{for any} \quad m=1,2,\dots,2^n,$$

where the numbers $\alpha_i^{(n)}(m)$ are defined by (3.4). Hence for any integer n we get

$$\max_{1\le m\le 2^n}\left\|\sum_{i=1}^{2^n}\left|\alpha_i^{(n)}(m)\right|\varphi_i\right\| \ge \frac{\max_{1\le m\le 2^n}\sum_{i=1}^{2^n}|\alpha_i^{(n)}(m)|}{2^{n+1}}\,\lambda(2^n). \tag{3.12}$$

We know that $\| \sum_{i=1}^{2^n} |\alpha_i^{(n)}(m)|\varphi_i\| = \varrho^{(n)}(m)$ and $\sum_{i=1}^{2^n} |\alpha_i^{(n)}(m)|$ is the value of $\varrho^{(n)}(m)$ in the space l_1 with respect to the natural basis. Therefore, by Theorem 3.5 we have

$$\max_{1\le m\le 2^n} \sum_{i=1}^{2^n} \left|\alpha_i^{(n)}(m)\right| = \frac{3n+7}{9}\, 2^n + (-1)^n\, \frac{2}{9} \quad \text{for any } n = 1, 2, \ldots .$$

Putting these expressions into (3.12) we complete the proof by elementary calculations. □

Remark 3.7. If a basis (φ_i) of a space X is in addition symmetric, then using the inequality of Proposition 2.2 (**A**) we can prove by analogy with Theorem 3.6 that

$$\varrho^{(n)} \ge \max\left\{\frac{n+2}{3}\,\lambda(2^n),\, 2^n\right\}, \qquad n = 1, 2, \ldots .$$

It follows from Theorem 3.6 that in spaces of type p, $p > 1$, for sufficiently large n the lower estimate 2^n is more precise than $\frac{n+2}{6}\,\lambda(2^n)$, because in such spaces we have $\lambda(2^n) \le T_p(X)\, 2^{n/p}$. Hence, the lower estimate $\frac{n+2}{6}\,\lambda(2^n)$ can compete with 2^n in spaces of type 1.

The following example shows that beside l_1 there exist Banach spaces different from l_1 with $\sup_n \varrho^{(n)}/2^n = \infty$.

Example 3.8. Consider the real function $f(t) = \frac{\sqrt{\log_2 5}}{5} \frac{t+4}{\sqrt{\log_2(t+4)}}$, $t \ge 1$. It is concave since for every $t \ge 1$ we have

$$f''(t) = \frac{\sqrt{\log_2 5}}{10\ln 2} \cdot \frac{-\log_2(t+4) + 3/(2\ln 2)}{(t+4)\log_2^{5/2}(t+4)} \le 0.$$

By Proposition 2.1(c) the sequence (λ_n) with $\lambda_n = f(n)$, $n = 1, 2, \ldots$, is concave. Therefore, there exists at least one Banach space X having a symmetric basis (φ_i) with symmetric constant equal to 1 such that $\lambda(n) = \| \sum_{i=1}^{n} \varphi_i\| = \lambda_n$ for every $n = 1, 2, \ldots$ (see [7], p. 120). Hence by Remark 3.7 for any integer n we have

$$\begin{aligned}\varrho^{(n)} \ge \frac{n+2}{3} \cdot \lambda(2^n) &= \frac{n+2}{3} \cdot \frac{\sqrt{\log_2 5}}{5} \cdot \frac{2^n+4}{\sqrt{\log_2(2^n+4)}} \\ &> \frac{\sqrt{\log_2 5}}{15} \cdot \frac{n+2}{\sqrt{n+2}} \cdot 2^n \ge \frac{\sqrt{\log_2 5}}{15} \cdot \sqrt{n+2} \cdot 2^n.\end{aligned}$$

The space X is not isomorphic to l_1 since

$$\lim_{n\to\infty}\sup_n \frac{\lambda(n)}{n} = \lim_{n\to\infty}\sup_n \left(\frac{\sqrt{\log_2 5}}{5}\cdot\frac{n+4}{n\sqrt{\log_2(n+4)}}\right) = 0.$$

In particular, it follows from the obtained estimate that the type of X does not exceed 1 (cf. Corollary 3.4).

4. Hadamard matrices

The main aim of this section is to clear up whether the estimates for Sylvester matrices found in Section 3 can be extended to general Hadamard matrices. Let $\mathcal{H}_n^{\mathrm{all}}$ be the set of all Hadamard matrices of order n, $n \in \mathbb{N}_{\mathcal{H}}$. For a Hadamard matrix $\mathcal{H}_n = [h_{ki}^n]$ we consider the same numerical characteristic $\varrho_{\mathcal{H}_n}(m) = \|\sum_{i=1}^n(\sum_{k=1}^m h_{ki}^n)\varphi_i\|$, $m = 1, 2, \dots, n$, where (φ_i) is a normalized basis of a Banach space X. Setting $a_k = \sum_{i=1}^n h_{ki}^n\varphi_i$, we notice that

$$\varrho_{\mathcal{H}_n}(m) = \left\|\sum_{k=1}^m a_k\right\|. \tag{4.1}$$

If (φ_i) is an unconditional basis with unconditional constant equal to 1, then we have $\max_{1\le m\le n}\varrho_{\mathcal{H}_n}(m) \le \lambda(n)\, n \le n^2$ for any $\mathcal{H}_n \in \mathcal{H}_n^{\mathrm{all}}$.

Finally we set $\varrho_{\mathcal{H}_n} = \max_{1\le m\le n}\varrho_{\mathcal{H}_n}(m)$ and $\varrho_n = \max_{\mathcal{H}_n\in\mathcal{H}_n^{\mathrm{all}}}\varrho_{\mathcal{H}_n}$.

Remark 4.1. Note that the characteristic $\varrho_{\mathcal{H}_n} = \varrho(\mathcal{H}_n)$ can be regarded as a norm of the Hadamard matrix $\mathcal{H}_n$. Indeed, let us denote by $\mathbf{M}_n$ the vector space of all square matrices of order n, $n \in \mathbb{N}_{\mathcal{H}}$, and let X be a Banach space with a basis (φ_i). One has $\mathcal{H}_n^{\mathrm{all}} \subset \mathbf{M}_n$. Let $\mathcal{T}_n = [t_{ki}^n] \in \mathbf{M}_n$ be a matrix and $\varrho(\mathcal{T}_n) = \max_{1\le m\le n}\|\sum_{i=1}^n(\sum_{k=1}^m t_{ki}^n)\varphi_i\|$. It is easy to see that ϱ is a norm in $\mathbf{M}_n$ and with respect to this norm $\mathbf{M}_n$ is a Banach space.

The following theorem gives us the lower estimate for ϱ_n.

Theorem 4.2. *Let X be a Banach space with a normalized unconditional basis whose unconditional constant is* 1. *Then we have*

$$\varrho_n \ge \max\left\{(1/\sqrt{2})\,\lambda(n)\sqrt{n},\, n\right\} \quad \textit{for any } n \in \mathbb{N}_{\mathcal{H}}.$$

Proof. If one of the columns of a Hadamard matrix $\mathcal{H}_n$ consists of $+1$ only, then we have $\varrho_{\mathcal{H}_n}(n) = n$ and the inequality $\varrho_n \geq n$ is evident.

Let $\mathcal{H}_n = [h^n_{ki}]$ be a Hadamard matrix of order n and $(r_k(t))_{k\leq n}$ be a sequence of Rademacher functions defined on the interval $[0,1]$. For every $t \in [0,1]$ the matrix $\mathcal{H}_{n,t} = [h^n_{ki}\, r_k(t)]$ is also a Hadamard matrix such that $\varrho_{\mathcal{H}_{n,t}} = \max_{1\leq m\leq n} \varrho_{\mathcal{H}_{n,t}}(m) = \max_{1\leq m\leq n} \| \sum_{k=1}^m a_k\, r_k(t)\|$, where $a_k = \sum_{i=1}^n h^n_{ki}\varphi_i$, $k = 1,2,\dots,n$.

Let $\xi(t) = \sum_{i=1}^n | \sum_{k=1}^n \langle \varphi_i^*, a_k\, r_k(t)\rangle | \varphi_i$. Using the fact that (φ_i) is an unconditional basis with unconditional constant equal to 1, it is easy to see that

$$\begin{aligned} \|\xi(t)\| &= \left\| \sum_{i=1}^n \sum_{k=1}^n \langle \varphi_i^*, a_k\rangle\, r_k(t)\varphi_i \right\| \\ &= \left\| \sum_{k=1}^n \left(\sum_{i=1}^n \langle \varphi_i^*, a_k\rangle\, \varphi_i \right) r_k(t) \right\| = \left\| \sum_{k=1}^n a_k\, r_k(t) \right\| \end{aligned}$$

for every $t \in [0,1]$. As the Rademacher functions are bounded, $\|\xi(t)\|$ is integrable with respect to the Lebesgue measure on $[0,1]$. Hence, there exists the Pettis integral $\mathbb{E}\,\xi$ of the measurable function ξ and $\mathbb{E}\,\|\xi\| \geq \|\mathbb{E}\,\xi\|$. It is easy to see that $\mathbb{E}\,\xi = \sum_{i=1}^n (\mathbb{E}\, | \sum_{k=1}^n \langle \varphi_i^*, a_k\rangle\, r_k(t)|)\varphi_i$.

As the Rademacher functions are bounded, $\varrho_{\mathcal{H}_{n,t}}$ is also integrable with respect to the Lebesgue measure on $[0,1]$, and using the Khintchine's inequality we have

$$\begin{aligned} \infty > \mathbb{E}\,\varrho_{\mathcal{H}_{n,t}} &= \mathbb{E} \max_{1\leq m\leq n} \left\| \sum_{k=1}^m a_k\, r_k(t) \right\| \geq \mathbb{E}\,\|\xi\| \geq \|\mathbb{E}\,\xi\| \\ &\geq (1/\sqrt{2}) \left\| \sum_{i=1}^n \left(\sum_{k=1}^n \langle \varphi_i^*, a_k\rangle^2 \right)^{1/2} \varphi_i \right\| = (1/\sqrt{2})\lambda(n)\,\sqrt{n}, \end{aligned}$$

where (φ_i^*) are the biorthogonal functionals associated to the basis (φ_i). Then, clearly, there exists a point $t_0 \in [0,1]$ such that $\varrho_{\mathcal{H}_{n,t_0}} \geq \mathbb{E}\,\varrho_{\mathcal{H}_{n,t}}$ and therefore $\varrho_n \geq \varrho_{\mathcal{H}_{n,t_0}} \geq (1/\sqrt{2})\lambda(n)\,\sqrt{n}$. □

An immediate consequence of this theorem is the following corollary.

Corollary 4.3. *In l_p, $1\leq p<2$, with the natural basis we have* $\sup_{n\in\mathbb{N}_{\mathcal{H}}} \varrho_n/n = \infty$.

For the spaces l_p the similar fact for the Sylvester matrices holds only for the space l_1 (see Theorem 3.5).

Let us estimate ϱ_n from above for the case of l_p, $1 \leq p < \infty$.

Theorem 4.4. *In l_p, $1 \le p < \infty$, with the natural basis for any $n \in \mathbb{N}_{\mathcal{H}}$ the following inequality holds*

$$\varrho_n \le \max\left\{n^{(p+2)/2p},\, n\right\}.$$

Proof. Let $p \ge 2$ and $\mathcal{H}_n \in \mathcal{H}_n^{all}$ be an arbitrary Hadamard matrix of order n. Using definition (4.1) and the fact that $\|a\|_{l_p} \le \|a\|_{l_2}$ one can see that

$$\varrho_{\mathcal{H}_n} \le \max_{1\le m\le n} \left\| \sum_{k=1}^{m} a_k \right\|_{l_2} = \max_{1\le m\le n} \left(\sum_{k=1}^{m} a_k, \sum_{k=1}^{m} a_k \right)^{1/2} = n,$$

where $(\cdot,\cdot)$ denotes the inner product in the space l_2. Hence in l_p, $p \ge 2$, the estimate $\varrho_n \le n$ holds.

Now let $1 \le p \le 2$ and $\mathcal{H}_n \in \mathcal{H}_n^{\mathrm{all}}$ be again an arbitrary Hadamard matrix of order n. If $a = (\alpha_i) \in l_p$ is a sequence of the length n (i.e. $\alpha_n \ne 0$ and $\alpha_i = 0$ for any $i > n$), then we have $\|a\|_{l_p} \le n^{(2-p)/2p}\,\|a\|_{l_2}$. Hence, we have

$$\varrho_{\mathcal{H}_n} \le n^{(2-p)/2p} \max_{1\le m\le n} \left\| \sum_{k=1}^{m} a_k \right\|_{l_2} = n^{(p+2)/2p}$$

and the theorem is proved. □

For Sylvester matrices Corollary 4.3 and Theorem 4.4 yield the following corollary.

Corollary 4.5. *Let $S^{(n)}$ be the Sylvester matrix of order 2^n, $n = 1, 2, \ldots$. Then in l_p, $p \ge 2$, with the natural basis we have*

$$\varrho^{(n)} = 2^n.$$

Theorem 4.2 and 4.4 imply the following assertion.

Corollary 4.6. *In l_p, $1 \le p \le \infty$, with respect to the natural basis for every $n \in \mathbb{N}_{\mathcal{H}}$ we have*

$$(1/\sqrt{2})\, n^{(p+2)/2p} \le \varrho_n \le n^{(p+2)/2p}, \quad \text{for } 1 \le p < 2,$$
$$\varrho_n = n, \quad \text{for } p \ge 2.$$

Let X be a Banach space (not necessarily with a basis), $x_1, x_2, \ldots, x_n$ be a sequence of elements from the unit ball of X and $\mathcal{H}_n \in \mathcal{H}_n^{\mathrm{all}}$, $n \in \mathbb{N}_{\mathcal{H}}$. Let us put $\hat{\varrho}_{\mathcal{H}_n}(m) = \|\sum_{i=1}^{n}(\sum_{k=1}^{m} h_{ki}^{n})x_i\|$, $m = 1, 2, \ldots, n$, $\hat{\varrho}_{\mathcal{H}_n} = \max_{1\le m\le n} \hat{\varrho}_{\mathcal{H}_n}(m)$ and $\hat{\varrho}_n = \max_{\mathcal{H}_n \in \mathcal{H}_n^{\mathrm{all}}} \hat{\varrho}_{\mathcal{H}_n}$.

Corollary 4.7. *For any $n \in \mathbb{N}_{\mathcal{H}}$ we have $\hat{\varrho}_n \leq n\sqrt{n}$.*

Proof. The proof goes analogously to the proof of Corollary 3.3. Following this way for the case $p = 1$ we use the estimate given by Corollary 4.6. □

Now we prove the analogue of Theorem 3.1 for the Hadamard matrices.

Theorem 4.8. *Let X be a Banach space with a normalized subsymmetric basis whose subsymmetric constant is 1. Then we have for any $n \in \mathbb{N}_{\mathcal{H}}$*

$$\varrho_n \leq \lambda([\sqrt{n}\,] + 1)\, n,$$

where $[\sqrt{n}\,]$ is the integer part of $\sqrt{n}$.

Proof. Let $\mathcal{H}_n = [h_{ki}^n]$ be a Hadamard matrix of order n. As we already have noted

$$\varrho_{\mathcal{H}_n} = \max_{1\leq m\leq n} \left\| \sum_{i=1}^{n} \left(\sum_{k=1}^{m} h_{ki}^n \right) \varphi_i \right\| \leq \max_{1\leq m\leq n} \sum_{i=1}^{n} \left| \sum_{k=1}^{m} h_{ki}^n \right| \leq n\sqrt{n} \tag{4.2}$$

for every $\mathcal{H}_n \in \mathcal{H}_n^{\mathrm{all}}$. For the sake of convenience let us introduce the notation

$$\alpha_i^{(n)}(m) = \left| \sum_{k=1}^{m} h_{ki}^n \right| \quad \text{for any } i, m = 1, 2, \ldots, n. \tag{4.3}$$

Using the definition of the Hadamard matrices and (4.2) we obtain the following properties of the numbers $\alpha_i^{(n)}(m)$:

(a). For all i and m the number $\alpha_i^{(n)}(m)$ is an integer and $0 \leq \alpha_i^{(n)}(m) \leq n$.

(b). For any m we have $\sum_{i=1}^{n} \alpha_i^{(n)}(m) \leq n\sqrt{n}$.

Denote by M the subset of X consisting of n points $\{\sum_{i=1}^{n} \alpha_i^{(n)}(m)\varphi_i : m = 1, 2, \ldots, n\}$, where $\alpha_i^{(n)}(m)$ is defined by (4.3). Then we have $\varrho_{\mathcal{H}_n} = \max_{x\in M} \|x\|$.

Let us consider the following subsets of X:

$$S = \left\{ \sum_{i=1}^{n} t_i\varphi_i : 0 \leq t_i \leq n,\ i = 1, 2, \ldots, n \right\} \quad \text{and} \quad T = \left\{ \sum_{i=1}^{n} t_i\varphi_i : \sum_{i=1}^{n} t_i \leq n\sqrt{n} \right\}.$$

Since S is an n-dimensional parallelepiped and T is a hyperplane in X, the sets S, T as well as their intersection $S \cap T$ are convex. Moreover, we have $M \subset S \cap T$. The set $S \cap T$ is compact because it is a bounded set in an n-dimensional subset of X spanned by the basis vectors $\varphi_1, \varphi_2, \ldots, \varphi_n$. According to the Krein–Milman theorem (see, for example, [3], p. 104) $S \cap T$ is a closed convex span of its extreme points. Hence we have

$$\varrho_{\mathcal{H}_n} = \max_{x \in M} \|x\| \le \sup_{x \in S \cap T} \|x\| = \sup_{x \in E} \|x\|, \tag{4.4}$$

where E is the set of all extreme points of $S \cap T$. The extreme points of the set S are the vertices of the parallelepiped S, i.e. the points of the form $\sum_{i=1}^{n} \beta_i \varphi_i$, where each β_i takes the values 0 or n. Since $E \subset S \cap T$, the set E contains those extreme points of S for which the condition $\sum_{i=1}^{n} \beta_i \le n\sqrt{n}$ is satisfied. If we denote by l the number of these β_i-s which are different from zero, then the last condition can be expressed as follows: $ln \le n\sqrt{n}$, or equivalently $l \le \sqrt{n}$. Since l is an integer, we get $l \le [\sqrt{n}\,]$. Since the basis (φ_i) is subsymmetric, the norm $\|\sum_{i=1}^{n} \beta_i \varphi_i\|$ can be estimated as follows $\|\sum_{i=1}^{n} \beta_i \varphi_i\| \le \lambda(l)\, n < \lambda([\sqrt{n}\,] + 1)\, n$.

It is easy to check that the set E, besides the vertices of the parallelepiped S, contains the points of the intersection of the bound of T with the edges of the parallelepiped S. The edges of S consists of the points which have the form $\sum_{i=1}^{n} \beta_i \varphi_i$, where one of β_i satisfies the condition $0 \le \beta_{i_0} \le n$ and all other β_i-s take the values 0 or n. Denote by l the number of β_i-s for which $\beta_i = n$. Due to the condition $\sum_{i=1}^{n} \beta_i \varphi_i \in T$, we have $\beta_{i_0} + ln \le n\sqrt{n}$. As $\beta_{i_0} \ge 0$ and l is an integer we have $l \le [\sqrt{n}\,]$. Since $0 \le \beta_{i_0} \le n$, using again that (φ_i) is a subsymmetric basis, we obtain

$$\Big\| \sum_{i=1}^{n} \beta_i \varphi_i \Big\| = \Big\| \beta_{i_0} \varphi_{i_0} + \sum_{i_0 \ne i = 1}^{n} \beta_i \varphi_i \Big\| \le \Big\| n \varphi_{i_0} + \sum_{i_0 \ne i = 1}^{n} \beta_i \varphi_i \Big\| \le \lambda([\sqrt{n}\,] + 1)\, n.$$

Thus, for every point x of the set E the estimate $\|x\| \le \lambda([\sqrt{n}\,] + 1)\, n$ is valid and using (4.4) we complete the proof of the theorem. □

Remark 4.9. We can rephrase Theorem 4.8 in the following way: Let $\mathcal{H}_n = [h_{ki}^n]$ be a Hadamard matrix of order $n \in \mathbb{N}_{\mathcal{H}}$ and let $a_k = \sum_{i=1}^{n} h_{ki}^n \varphi_i$, $k = 1, 2, \ldots, n$, where (φ_i) is a normalized subsymmetric basis of a Banach space X with subsymmetric constant equal to 1. Then we have

$$\max_{1 \le m \le n} \Big\| \sum_{k=1}^{m} \vartheta_k a_k \Big\| \le \lambda\Big([\sqrt{n}\,] + 1 \Big)\, n$$

for every sign $\vartheta_k \in \{-1, 1\}$, $k = 1, 2, \ldots, n$, every Hadamard matrix $\mathcal{H}_n \in \mathcal{H}_n^{\mathrm{all}}$ and every positive integer $n \in \mathbb{N}_{\mathcal{H}}$.

By Theorem 3.1, in a Banach space with a normalized subsymmetric basis whose subsymmetric constant is 1 we have $\varrho^{(n)}/(n \cdot 2^n) \le 1$. On the other hand, by Theorem 3.5 in the space l_1 we have $\varrho^{(n)}/(n \cdot 2^n) \ge 1/3$. Using Sylvester and Hadamard matrices we can characterize the spaces isomorphic to l_1 as follows.

Theorem 4.10. *Let X be a Banach space with a normalized subsymmetric basis (φ_i) whose subsymmetric constant is 1. The following statements are equivalent:*

(i). *There is a constant $\delta > 0$ such that $\varrho_n/(n\sqrt{n}) \geq \delta$ for every $n \in \mathbb{N}_{\mathcal{H}}$, where δ is independent of n.*

(ii). *X is isomorphic to l_1.*

(iii). *There exists a constant $\varepsilon > 0$ which does not depend on n such that for every $n = 1, 2, \dots$ we have $\varrho^{(n)}/(n \cdot 2^n) \geq \varepsilon$.*

Proof. (i) $\Longrightarrow$ (ii). Using Theorem 4.8 for every $n \in \mathbb{N}_{\mathcal{H}}$ we have

$$0 < \delta \leq \varrho_n/(n\sqrt{n}\,) \leq \lambda([\sqrt{n}\,]+1)\,n/(n\sqrt{n}\,) = \lambda([\sqrt{n}\,]+1)/\sqrt{n}.$$

Therefore one has $\lambda([\sqrt{n}\,])/\sqrt{n} \geq \delta/2 > 0$ for infinitely many n. Now the validity of the statement (ii) follows from the fact which was mentioned in Section 2: if

$$\lim_{n\to\infty} \sup \lambda(n)/n > 0,$$

then X is isomorphic to l_1.

(ii) $\Longrightarrow$ (iii). Let X be isomorphic to l_1, and denote by $T : X \to l_1$ an isomorphism between X and l_1. It is clear that $(T\varphi_i)$ is an unconditional basis in l_1. Since in l_1 all normalized unconditional bases are equivalent (see [7], p. 71), there exists a bounded linear operator $S : l_1 \to l_1$ with bounded inverse operator, such that $T\varphi_i = Se_i$ for every integer i, where (e_i) is a sequence of the unit vectors in l_1. By Theorem 3.5 for every integer n we have

$$1/3 \leq \max_{1\leq m\leq 2^n} \left\| \sum_{i=1}^{2^n} \left| \sum_{k=1}^{m} s_{ki}^{(n)} \right| e_i \right\| / \left(n \cdot 2^n\right)$$

$$= \max_{1\leq m\leq 2^n} \left\| \sum_{i=1}^{2^n} \left| \sum_{k=1}^{m} s_{ki}^{(n)} \right| S^{-1}T\varphi_i \right\| / \left(n \cdot 2^n\right) \leq \|S^{-1}T\| \max_{1\leq m\leq 2^n} \varrho^{(n)}(m)/\left(n \cdot 2^n\right).$$

With $\varepsilon = 1/(3\|S^{-1}T\|) > 0$ we get the validity of assertion (iii).

The implication (iii) $\Longrightarrow$ (i) is true because $2^n \in \mathbb{N}_{\mathcal{H}}$. □

5. Unsolved problem

Let (e_i) be the natural basis of the space l_1, $\mathcal{S}^{(n)} = [s_{ki}^{(n)}]$ be the Sylvester matrix of order 2^n, $n = 1, 2, \dots$, and $(a_k)_{k\leq 2^n}$ be the sequence in l_1 defined by

$$a_k = \sum_{i=1}^{2^n} s_{ki}^{(n)} e_i, \quad k = 1, 2, \dots, 2^n.$$

Let us formulate the assertion of Theorem 3.5 in the following manner:

$$\varrho^{(n)} = \left\| \sum_{k=1}^{m_n} a_k \right\|_{l_1} = (3n+7)2^n/9 + 2(-1)^n/9,$$

where $m_n = (2^{n+1} + (-1)^n)/3$.

Now let us consider a permutation $\sigma : \{1, 2, \ldots, 2^n\} \to \{1, 2, \ldots, 2^n\}$ and the following expression:

$$\left\| \sum_{k=1}^{m_n} a_{\sigma(k)} \right\|_{l_1}.$$

By Corollary 4.6 for every permutation $\sigma : \{1, 2, \ldots, 2^n\} \to \{1, 2, \ldots, 2^n\}$ we have

$$\left\| \sum_{k=1}^{m_n} a_{\sigma(k)} \right\|_{l_1} \leq 2^{3n/2}.$$

The authors do not know yet the answer for the following conjecture:

Conjecture 5.1. For any positive integer n and for any permutation $\sigma : \{1, 2, \ldots, 2^n\} \to \{1, 2, \ldots, 2^n\}$ the following inequality holds:

$$\left\| \sum_{k=1}^{m_n} a_{\sigma(k)} \right\|_{l_1} \geq (3n+7)2^n/9 + 2(-1)^n/9.$$

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