Journal of Lie Theory
Volume **25** (2015) ??–??
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Version of November 30, 2014

Three-Dimensional Topological Loops with Nilpotent Multiplication Groups

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Abstract. In this paper we describe the structure of indecomposable nilpotent Lie groups which are multiplication groups of three-dimensional simply connected topological loops. In contrast to the 2-dimensional loops there is no connected topological loop of dimension ≥ 3 such that the Lie algebra of its multiplication group is an elementary filiform Lie algebra. We determine the indecomposable nilpotent Lie groups of dimension ≤ 6 and their subgroups which are the multiplication groups and the inner mapping groups of the investigated loops. We prove that all multiplication groups have 1-dimensional centre and the corresponding loops are centrally nilpotent of class 2.
Mathematics Subject Classification 2000: 57S20, 22E25, 20N05, 57M60, 22F30.
Key Words and Phrases: Multiplication group of loops, topological transformation group, nilpotent Lie group.

1. Introduction

The multiplication group $Mult(L)$ is an essential tool for the research in loop theory since the structure of the normal subgroups of $Mult(L)$ reflects the structure of the normal subloops of L (cf. [1], [2], [3]). In [14] the authors established necessary and sufficient conditions for a group K to be the multiplication group of L. These criterions say that one can use special transversals A and B with respect to a subgroup S of K. In these conditions the subgroup S is the stabilizer of the identity of L in $Mult(L)$ and it is called the inner mapping group $Inn(L)$ of L. Furthermore, the transversals A and B correspond to the sets of left and right translations of L, respectively. These criterions have been successfully applied for finite loops (cf. [1], [2], [13]-[16]) and they can be also used effectively for connected topological loops L having a Lie group as their multiplication group (cf. [4]-[6]).

In [4] we proved that only the elementary filiform Lie groups $\mathcal{F}_n$, $n \geq 4$, are the multiplication groups $Mult(L)$ of 2-dimensional connected simply connected topological loops L. In contrast to this, Proposition 2.8 shows that there does not exist any connected topological loop L of dimension ≥ 3 having a Lie group with elementary filiform Lie algebra as the group $Mult(L)$. In [5] and [6] we analyzed

* This paper is supported by the János Bolyai Research Fellowship and by the European Union's Seventh Framework Programme (FP7/2007-2013) under grant agreements no. 317721, no. 318202.

the question under which circumstances will be nilpotent or solvable Lie group the multiplication group of a topological loop L of dimension ≤ 3. Till now this characterization has been resulted only decomposable nilpotent and solvable Lie groups as the group $Mult(L)$ of 3-dimensional topological loops with the exception of the direct product $\mathcal{F}_n \times_Z \mathcal{F}_m$ with amalgamated centre Z, $n, m \ge 3$. In [6] it is also proved that all 3-dimensional simply connected topological loops L having a solvable, non-nilpotent Lie group of dimension ≤ 5 as their multiplication group are centrally nilpotent of class 2.

In this paper we focused our attention on the study of indecomposable nilpotent Lie groups as well as on the search for 3-dimensional centrally nilpotent topological loops of class 3. In Propositions 2.6 and 2.7 we give the precise structure of the 3-dimensional connected simply connected topological loops L and those of the indecomposable nilpotent Lie groups which are the groups $Mult(L)$ of L. In particular the centrally nilpotent loops of class 3 are characterized by Proposition 2.7 b). All groups $Mult(L)$ have 1-dimensional centre (cf. Proposition 2.9).

In Theorems 3.1 and 4.3 we list the indecomposable nilpotent Lie groups of dimension ≤ 6 which are the groups $Mult(L)$ of 3-dimensional connected simply connected topological loops L and the subgroups of $Mult(L)$ which are the inner mapping groups $Inn(L)$ of L. None of the 4-dimensional nilpotent Lie groups is represented as the group $Mult(L)$ for 3-dimensional loops L. Summarizing these results we have:

Theorem *Let L be a 3-dimensional connected simply connected topological proper loop such that the group $Mult(L)$ is an indecomposable nilpotent Lie group. Then, one has $dim(Mult(L)) \ge 5$.*
a) $Mult(L)$ has dimension 5 if and only if it is either the direct product $\mathcal{F}_3 \times_Z \mathcal{F}_3$ with amalgamated centre Z or the simply connected Lie group with 1-dimensional centre and 2-dimensional commutator subgroup.
b) $Mult(L)$ has dimension 6 if and only if it is either the direct product $\mathcal{F}_4 \times_Z \mathcal{F}_3$ with amalgamated centre Z or a simply connected Lie group with 1-dimensional centre and 3-dimensional commutator subgroup.
In all cases the loops L are centrally nilpotent of class 2.

Together with the main result in [6] we obtain that all 3-dimensional connected simply connected topological loops L having an at most 5-dimensional Lie group as their multiplication group are centrally nilpotent of class 2.

2. Preliminaries

A loop is a binary system $(L, \cdot)$ if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution, which we denote by $y = a \backslash b$ and $x = b/a$. A loop L is proper if it is not a group.
The left and right translations $\lambda_a = y \mapsto a \cdot y : L \to L$ and $\rho_a : y \mapsto y \cdot a : L \to L$, $a \in L$, are permutations of L.
The permutation group $Mult(L) = \langle \lambda_a, \rho_a;\ a \in L \rangle$ is called the multiplication

group of L. The stabilizer of the identity element $e \in L$ in $Mult(L)$ is denoted by $Inn(L)$, and $Inn(L)$ is called the inner mapping group of L.

Let K be a group, let $S \leq K$, and let A and B be two left transversals to S in K. We say that A and B are S-connected if $a^{-1}b^{-1}ab \in S$ for every $a \in A$ and $b \in B$. The core $Co_K(S)$ of S in K is the largest normal subgroup of K contained in S. If L is a loop, then $\Lambda(L) = \{\lambda_a;\ a \in L\}$ and $R(L) = \{\rho_a;\ a \in L\}$ are $Inn(L)$-connected transversals in the group $Mult(L)$ and the core of $Inn(L)$ in $Mult(L)$ is trivial. We often use the following (see [14], Theorem 4.1 and Proposition 2.7).

Lemma 2.1. *A group K is isomorphic to the multiplication group of a loop if and only if there exists a subgroup S with $Co_K(S) = 1$ and S-connected transversals A and B satisfying $K = \langle A, B \rangle$.*

Lemma 2.2. *Let L be a loop with multiplication group $Mult(L)$ and inner mapping group $Inn(L)$. Then the normalizer $N_{Mult(L)}(Inn(L))$ is the direct product $Inn(L) \times Z(Mult(L))$, where $Z(Mult(L))$ is the centre of the group $Mult(L)$.*

The kernel of a homomorphism $\alpha : (L, \cdot) \to (L', *)$ of a loop L into a loop L' is a normal subloop N of L, i.e. a subloop of L such that

$$x \cdot N = N \cdot x,\ (x \cdot N) \cdot y = x \cdot (N \cdot y),\ x \cdot (y \cdot N) = (x \cdot y) \cdot N. \qquad (1)$$

The centre $Z(L)$ of a loop L consists of all elements z which satisfy the equations $zx \cdot y = z \cdot xy,\ x \cdot yz = xy \cdot z,\ xz \cdot y = x \cdot zy,\ zx = xz$ for all $x, y \in L$. If we put $Z_0 = e$, $Z_1 = Z(L)$ and $Z_i/Z_{i-1} = Z(L/Z_{i-1})$, then we obtain a series of normal subloops of L. If Z_{n-1} is a proper subloop of L but $Z_n = L$, then L is centrally nilpotent of class n.

Lemma 2.3. *Let L be a loop with multiplication group $Mult(L)$ and identity element e.*
(i) Let α be a homomorphism of the loop L onto the loop $\alpha(L)$ with kernel N. Then α induces a homomorphism of the group $Mult(L)$ onto the group $Mult(\alpha(L))$. Let $M(N)$ be the set $\{m \in Mult(L);\ xN = m(x)N \text{ for all } x \in L\}$. Then $M(N)$ is a normal subgroup of $Mult(L)$ containing the multiplication group $Mult(N)$ of N and the multiplication group of the factor loop L/N is isomorphic to $Mult(L)/M(N)$.
(ii) For every normal subgroup $\mathcal{N}$ of $Mult(L)$ the orbit $\mathcal{N}(e)$ is a normal subloop of L and $\mathcal{N} \leq M(\mathcal{N}(e))$.

Proof. The main part of the assertion was proved by Albert in [1], Theorems 3, 4 and 5 and by Bruck in [3], IV.1, Lemma 1.3. It remains only to show that the multiplication group $Mult(N)$ of the normal subloop N is contained in the group $M(N)$. It is enough to prove that the transformations λ_y, ρ_y, λ_y^{-1}, ρ_y^{-1} for all $y \in N$ are elements of $M(N)$. By (1) for every $y \in N$ there exists an $y' \in N$ such that for all $x \in L$ one has $\lambda_y(x)N = (y \cdot x)N = (x \cdot y')N = x(y'N) = xN$, $\rho_y(x)N = (x \cdot y)N = x(yN) = xN$. For all $y \in N$, $x \in L$ one has $\lambda_y^{-1}(x)N =$

$(y\backslash x)N = xN$ if and only if there exists an element $y' \in N$ such that $y\backslash x = x \cdot y'$ or equivalently $x = y\cdot(x\cdot y')$. By the normality of N there are elements $y'', y''' \in N$ such that $x = (x \cdot y') \cdot y'' = x(y' \cdot y''')$. Hence with $y' = e/y'''$ the last identity is true. For all $y \in N$, $x \in L$ one has $\rho_y^{-1}(x)N = (x/y)N = xN$ precisely if x/y is an element of xN, i. e. for suitable $y' \in N$ we have $x/y = x \cdot y'$ or equivalently $x = (x \cdot y') \cdot y$. Since N is normal there is $\tilde{y} \in N$ such that $x = x \cdot (y' \cdot \tilde{y})$. With $y' = e/\tilde{y}$ the last identity is true. This proves the assertion. ■

A loop L is called topological if L is a topological space and the binary operations $(x, y) \mapsto x \cdot y,\ (x, y) \mapsto x\backslash y, (x, y) \mapsto y/x : L \times L \to L$ are continuous. Let G be a connected Lie group, let H be a subgroup of G. A section $\sigma : G/H \to G$ is called sharply transitive, if the set $\sigma(G/H)$ operates sharply transitively on G/H, which means that for any xH and yH there exists precisely one $z \in \sigma(G/H)$ with $zxH = yH$. Every connected topological loop L having a Lie group G as the group topologically generated by the left translations of L is obtained on a homogeneous space G/H, where H is a closed subgroup of G with $Co_G(H) = 1$ and $\sigma : G/H \to G$ is a continuous sharply transitive section such that $\sigma(H) = 1 \in G$ and the subset $\sigma(G/H)$ generates G. The multiplication of L on the manifold G/H is defined by $xH * yH = \sigma(xH)yH$ and the group G is the group topologically generated by the left translations of L. Moreover, the subgroup H is the stabilizer of the identity element $e \in L$ in the group G.

Lemma 2.4. *For any connected topological loop there is a universal covering loop. Any 3-dimensional proper connected simply connected topological loop having a solvable Lie group as its multiplication group is homeomorphic to $\mathbb{R}^3$.*

Lemma 2.4 is proved in [11], IX.1 and in [5], Lemma 3.3, p. 390.
The elementary filiform Lie group $\mathcal{F}_n$ is the simply connected Lie group of dimension $n \geq 3$ such that its Lie algebra has a basis $\{e_1, \cdots, e_n\}$ with $[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq n-1$. A 2-dimensional simply connected topological loop $L_{\mathcal{F}}$ is called an elementary filiform loop if its multiplication group is an elementary filiform Lie group $\mathcal{F}_n$, $n \geq 4$ ([5]). A Lie group is called indecomposable if its Lie algebra is indecomposable, i.e. it is not the direct sum of two proper ideals.
Now we collect the known results about nilpotent Lie groups which are the multiplication groups of 3-dimensional topological loops (cf. Lemmata 3.4, 3.5, 3.6 and Propositions 3.7, 3.8, 4.3 in [5], pp. 390-394, Theorem 11 in [1], Theorem 6 in [6]).

Lemma 2.5. *Let L be a 3-dimensional proper connected simply connected topological loop such that its multiplication group $Mult(L)$ is a nilpotent Lie group.*
a) Then the centre Z of the group $Mult(L)$ and the centre $Z(L) = Z(e)$ of the loop L are isomorphic to the group $\mathbb{R}^n$, $n = \{1, 2\}$, where e is the identity of L.
b) Every 1-dimensional normal subloop N of L is a central subgroup of L.
c) The loop L is centrally nilpotent. Moreover, L is an extension of a 2-dimensional centrally nilpotent loop M by the group $\mathbb{R}$ and also an extension of the group $N = \mathbb{R}$ by a 2-dimensional centrally nilpotent loop K such that N is a normal subloop of M.
d) If dim $Z(L) = 1$ and the factor loop $L/Z(L)$ is isomorphic to the group $\mathbb{R}^2$ or

if dim $Z(L) = 2$*, then* L *is centrally nilpotent of class* 2 *and the inner mapping group* $Inn(L)$ *of the loop* L *is abelian.*
e) If dim $Z(L) = 2$*, then* $Mult(L)$ *is a semidirect product of a group* $Q \cong \mathbb{R}$ *by the abelian normal subgroup* $M \cong \mathbb{R}^m$*,* $m \geq 3$*, such that* $M = Z \times Inn(L)$*, where* $\mathbb{R}^2 = Z \cong Z(L)$ *is the centre of* $Mult(L)$*.*
f) Let N *be a* 1*-dimensional central subgroup of* $Mult(L)$*, then the orbit* $N(e) \cong \mathbb{R}$ *is a central subgroup of* L *and we have the following possibilities:*
If the factor loop $L/N(e)$ *is isomorphic to the abelian group* $\mathbb{R}^2$*, then* $Mult(L)$ *is a semidirect product of a group* $Q \cong \mathbb{R}^2$ *by the abelian normal subgroup* $P \cong \mathbb{R}^m$*,* $m \geq 2$*, such that* $P = N \times Inn(L)$*.*
If the factor loop $L/N(e)$ *is isomorphic to a* 2*-dimensional elementary filiform loop* $L_{\mathcal{F}}$*, then there is a normal subgroup* S *of the group* $Mult(L)$ *containing* N *such that the factor group* $Mult(L)/S$ *is an elementary filiform Lie group* $\mathcal{F}_n$ *with* $n \geq 4$*.*
g) The unique 4*-dimensional indecomposable nilpotent Lie group* $\mathcal{F}_4$ *is not the multiplication group of a* 3*-dimensional topological loop.*

In the next Propositions we describe the indecomposable nilpotent Lie groups which are multiplication groups of 3-dimensional topological loops.

Proposition 2.6. *Let* L *be a* 3*-dimensional proper connected simply connected topological loop such that its multiplication group* $Mult(L)$ *is an indecomposable nilpotent Lie group and the centre* Z *of* $Mult(L)$ *has dimension* 2*. Then* L *is centrally nilpotent of class* 2*. The group* $Mult(L)$ *has dimension at least* 5 *and it is a semidirect product of a group* $Q \cong \mathbb{R}$ *by the abelian group* $M = Z \times Inn(L) \cong \mathbb{R}^m$*,* $m \geq 4$*, where* $\mathbb{R}^2 = Z \cong Z(L)$*. For every* 1*-dimensional connected subgroup* N *of* Z *the orbit* $N(e)$ *is a* 1*-dimensional connected central subgroup of* L *and the factor loop* $L/N(e)$ *is isomorphic to an elementary filiform loop* $L_{\mathcal{F}}$*. The group* $Mult(L)$ *has a normal subgroup* S *containing* $N \cong \mathbb{R}$ *such that the factor group* $Mult(L)/S$ *is an elementary filiform Lie group* $\mathcal{F}_n$ *with* $n \geq 4$*.*

Proof. By Lemma 2.5 d), e), g) one has L is centrally nilpotent of class 2, $\dim(Mult(L)) \geq 5$ and the group $Mult(L)$ is a semidirect product as in the assertion.

The latter part of the assertion is proved in Lemma 2.5 f) if we show that the factor loop $L/N(e)$ cannot be isomorphic to the group $\mathbb{R}^2$.

If the factor loop $L/N(e)$ would be isomorphic to the group $\mathbb{R}^2$, then by Lemma 2.5 f) the group $Mult(L)$ is a semidirect product of a group $Q \cong \mathbb{R}^2$ by the abelian group $P = N \times Inn(L)$. As $Mult(L)/P$ is isomorphic to $\mathbb{R}^2$ the commutator subgroup $Mult(L)'$ is contained in P. Since the Lie group $Mult(L)$ is indecomposable Z is a subgroup of $Mult(L)'$ and therefore Z is a subgroup of P. As $Co_{Mult(L)}(Inn(L)) = \{1\}$ one has $Inn(L) \cap Z = \{1\}$. Hence P contains only a proper subgroup N of Z which is a contradiction. ∎

Proposition 2.7. *Let* L *be a* 3*-dimensional proper connected simply connected topological loop such that its multiplication group* $Mult(L)$ *is an indecomposable*

nilpotent Lie group with 1-dimensional centre Z. Then we get $Z(e) = Z(L) \cong \mathbb{R}$ and the group $Mult(L)$ has dimension at least 5. Moreover, one of the following possibilities holds:
a) L is centrally nilpotent of class 2. Then the factor loop $L/Z(L)$ is isomorphic to $\mathbb{R}^2$ and the group $Mult(L)$ is a semidirect product of a group $Q \cong \mathbb{R}^2$ by the abelian group $P = Z \times Inn(L) \cong \mathbb{R}^m$, $m \geq 3$.
b) L is centrally nilpotent of class 3. Then the factor loop $L/Z(L)$ is isomorphic to an elementary filiform loop $L_{\mathcal{F}}$ and the group $Mult(L)$ has a normal subgroup S containing $Z \cong \mathbb{R}$ such that the orbit $S(e)$ is the centre of L, S induces the sharply transitive group $\mathbb{R}$ on $S(e)$ and the factor group $Mult(L)/S$ is an elementary filiform Lie group $\mathcal{F}_n$ with $n \geq 4$. The loop L has a normal subloop M isomorphic either to $\mathbb{R}^2$ or to an elementary filiform loop $L_{\mathcal{F}}$ such that $Z(e)$ is a central subgroup of M, L/M is isomorphic to $\mathbb{R}$ and the factor loop $M/Z(e)$ coincides with $Z(L/Z(e))$. Moreover, the group $Mult(L)$ has a normal subgroup V such that the orbit $V(e)$ is the loop M, V induces an elementary filiform group $\mathcal{F}_n$ with $n \geq 3$ on $M(e)$, $Mult(L)/V \cong \mathbb{R}$, V contains the inner mapping group $Inn(L)$ of L as well as the group $Mult(M)$ of M and Z is a central subgroup of $Mult(M)$. The centre of the factor group $Mult(L)/S$ coincides with the multiplication group of the factor loop $M/Z(e)$.

Proof. Lemma 2.5 g) yields that the group $Mult(L)$ has dimension at least 5.

Applying Lemma 2.5 f) to the centre $Z \cong \mathbb{R}$ of $Mult(L)$ we obtain assertion a) and that the group $Mult(L)$ has a normal subgroup S as in assertion b). Moreover, by Lemma 2.5 b) and c) the loop L has an upper central series $e < Z(e) < M < L$, where the normal subloop M of L is isomorphic either to $\mathbb{R}^2$ or to a loop $L_{\mathcal{F}}$ such that $L/M \cong \mathbb{R}$, $Z(e) = Z(L)$ is a central subgroup of M and $Z(L/Z(e)) = M/Z(e)$. Since M is normal in L by Lemma 2.3 the group $Mult(L)$ has a normal subgroup $V = \{v \in Mult(L); xM = v(x)M \text{ for all } x \in L\}$. Hence one has $V(e) = M$, $Mult(L)/V \cong \mathbb{R}$ and V contains the multiplication group $Mult(M)$ of M. Since the multiplication group of $Z(L)$ is Z it is a central subgroup of $Mult(M) < Mult(L)$. The group $Mult(L)/V$ operates sharply transitively on the orbits of M in L. Hence the inner mapping group $Inn(L)$ is a subgroup of V. The group V induces on the orbit $M(e)$ which is homeomorphic to $\mathbb{R}^2$ either the sharply transitive group $\mathbb{R}^2$ or an elementary filiform group $\mathcal{F}_n$, $n \geq 3$. This means the group V has a normal subgroup N such that $N(e) = e$ and V/N is isomorphic either to $\mathbb{R}^2$ or to $\mathcal{F}_n$, $n \geq 3$. If the factor group V/N is abelian, then N contains the commutator subgroup V'. As the centre Z of $Mult(L)$ is contained in V' we get a contradiction to the fact that $Z(e) = Z(L) \neq \{e\}$. Therefore V does not induce on $M(e)$ the group $\mathbb{R}^2$. The multiplication group of the factor loop $L/Z(e)$ is the factor group $Mult(L)/S$. The multiplication group of $Z(L/Z(e))$ coincides with the centre $Z(Mult(L)/S)$ of factor group $Mult(L)/S$. As $M/Z(e) = Z(L/Z(e))$ the group $Mult(M/Z(e))$ coincides with $Z(Mult(L)/S)$. ■

In contrast to the 2-dimensional loops we prove that the Lie groups with elementary filiform Lie algebra cannot be the multiplication group of connected topological loops of dimension ≥ 3.

Proposition 2.8. *There does not exist any connected topological loop of dimension ≥ 3 such that its multiplication group is a Lie group with elementary filiform Lie algebra.*

Proof. Assume that there exists a connected topological loop L such that $\dim(L) = m \geq 3$ with the required properties. Denote by $\{e_1, e_2, \cdots, e_n\}$ with $[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq n-1$ the basis of the Lie algebra $\mathbf{f_n}$ of $\mathcal{F}_n$. Then the centre $\mathbf{z}(\mathbf{f_n})$ of $\mathbf{f_n}$ is the subalgebra $\langle e_n \rangle$. First we assume that the inner mapping group $Inn(L)$ of L is abelian. The Lie algebra of the maximal abelian subgroup of $\mathcal{F}_n$ is the ideal $\mathbf{i} = \langle e_2, \cdots, e_n \rangle$ of the Lie algebra $\mathbf{f_n}$. As $Co_{\mathcal{F}_n}(Inn(L)) = \{1\}$ (cf. Lemma 2.1) the Lie algebra $\mathbf{inn(L)}$ is a proper subalgebra with codimension $m-1$ of $\mathbf{i}$ which does not contain the centre $\mathbf{z}(\mathbf{f_n})$ of $\mathbf{f_n}$. The Lie group I of the ideal $\mathbf{i}$ is the normalizer of the Lie group $Inn(L)$ of $\mathbf{inn(L)}$. But I contains as proper subgroup the direct product $Z \times Inn(L)$, where Z is the Lie group of the centre $\mathbf{z}(\mathbf{f_n})$. This contradicts Lemma 2.2.

Now we suppose that the inner mapping group $Inn(L)$ of L is non-abelian. In Lemma 8 in [5], pp. 424-425, we have shown that any non-abelian subalgebra of the elementary filiform Lie algebra $\mathbf{f_n}$ must contain the centre $\mathbf{z}(\mathbf{f_n})$ of $\mathbf{f_n}$. This is a contradiction to the fact that $Co_{\mathcal{F}_n}(Inn(L)) = \{1\}$. Hence the assertion is proved. ∎

Applying Proposition 2.6 and the list of the indecomposable nilpotent Lie algebras of dimension 5 and 6 ([7], pp. 167-168, [10], pp. 646-647) we get the following:

Proposition 2.9. *If L is a 3-dimensional connected topological proper loop having an at most 6-dimensional indecomposable nilpotent Lie group as its multiplication group, then L has 1-dimensional centre.*

Proof. By Lemma 2.4 we may assume that L is homeomorphic to $\mathbb{R}^3$. We show that the indecomposable nilpotent Lie algebras of dimension ≤ 6 having 2-dimensional centre cannot be the Lie algebra of the multiplication group of L. Among the 5- and the 6-dimensional indecomposable nilpotent Lie algebras only the Lie algebras $\mathbf{g}_{5.1}$, $\mathbf{g}_{5.3}$, $L_{6,19}(\epsilon)$, $L_{6,21}(\epsilon)$ with $\epsilon = 0$ and $L_{6,i}$ for $i = 22, 23, 24, 25$ have 2-dimensional centre (cf. [7], [10]). The Lie algebras $\mathbf{g}_{5.3}$, $L_{6,19}(\epsilon)$, $L_{6,21}(\epsilon)$ with $\epsilon = 0$ and $L_{6,i}$ for $i = 22, 23, 24$ has no abelian ideal of codimension 1. Hence these Lie algebras are excluded (cf. Proposition 2.6).
Assume that the Lie algebra of the multiplication group $Mult(L)$ of L is either the Lie algebra $\mathbf{g}_{5.1}$ or $L_{6,25}$. The centre of $\mathbf{g}_{5.1}$ is $\mathbf{z}(g_{5.1}) = \langle e_1, e_2 \rangle$, respectively of $L_{6,25}$ is $\mathbf{z}(L_{6,25}) = \langle e_5, e_6 \rangle$. Let $\mathbf{n_1}$ be the subalgebra $\langle e_1 \rangle$ of $\mathbf{z}(g_{5.1})$, respectively $\mathbf{n_2}$ be the subalgebra $\langle e_5 \rangle$ of $\mathbf{z}(L_{6,25})$. Since there does not exist an ideal $\mathbf{s_1}$ of $\mathbf{g}_{5.1}$ containing $\mathbf{n_1}$ and an ideal $\mathbf{s_2}$ of $L_{6,25}$ containing $\mathbf{n_2}$ such that the factor Lie algebra $\mathbf{g}_{5.1}/\mathbf{s_1}$, respectively $L_{6,25}/\mathbf{s_2}$ is an elementary filiform Lie algebra of dimension ≥ 4 we get a contradiction to Proposition 2.6. This proves the assertion. ∎

3. Five-dimensional nilpotent Lie groups which are multiplication groups of three-dimensional loops

Theorem 3.1. *Let L be a connected simply connected topological proper loop of dimension 3 such that its multiplication group $Mult(L)$ is a 5-dimensional indecomposable nilpotent Lie group. Then the centre $Z(L)$ of L is isomorphic to $\mathbb{R}$ and the factor loop $L/Z(L)$ is isomorphic to $\mathbb{R}^2$. Moreover, the following Lie groups are the multiplication groups $Mult(L)$ and the following subgroups are the inner mapping groups $Inn(L)$ of L:*
1) $Mult(L)_1$ is the direct product $\mathcal{F}_3 \times_Z \mathcal{F}_3$ with amalgamated centre Z. It is represented on $\mathbb{R}^5$ by the multiplication

$$g(q_1, z_1, w_1, x_1, y_1)g(q_2, z_2, w_2, x_2, y_2) =$$

$$g(q_1 + q_2 + z_1x_2 + w_1y_2, z_1 + z_2, w_1 + w_2, x_1 + x_2, y_1 + y_2).$$

$Inn(L)$ is the subgroup $\{g(0, z, w, 0, 0), z, w \in \mathbb{R}\}$.
2) $Mult(L)_2$ is the unique 5-dimensional simply connected indecomposable nilpotent Lie group with 1-dimensional centre and 2-dimensional commutator subgroup. It is represented on $\mathbb{R}^5$ by the multiplication

$$g(x_1, y_1, q_1, z_1, w_1)g(x_2, y_2, q_2, z_2, w_2) =$$

$$g(x_1 + x_2 + q_1z_2 + w_1y_2 + \frac{w_1^2q_2}{2}, y_1 + y_2 + w_1q_2, q_1 + q_2, z_1 + z_2, w_1 + w_2).$$

$Inn(L)_2$ is one of the groups $Inn(L)_{2,1} = \{g(0, y, q, 0, 0), y, q \in \mathbb{R}\}$, $Inn(L)_{2,2} = \{g(0, y, 0, z, 0), y, z \in \mathbb{R}\}$.

Proof. By Lemma 2.4 we may assume that L is homeomorphic to $\mathbb{R}^3$. According to Propositions 2.8, 2.9 and to the list of [7], pp. 167-168, the Lie algebra of the group $Mult(L)$ of L can be only one of the following Lie algebras:
$\mathbf{g}_{5.4}$: $[e_2, e_4] = e_1, [e_3, e_5] = e_1$,
$\mathbf{g}_{5.5}$: $[e_3, e_4] = e_1, [e_2, e_5] = e_1, [e_3, e_5] = e_2$,
$\mathbf{g}_{5.6}$: $[e_3, e_4] = e_1, [e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3$.
According to the list of [7], pp. 167-168, among the 5-dimensional indecomposable nilpotent Lie algebras only the Lie algebra $\mathbf{g}_{5.5}$ has 1-dimensional centre and 2-dimensional commutator ideal. A linear representation of the simply connected Lie group $G_{5.4}$ is given in assertion 1) and that of the simply connected Lie group $G_{5.5}$ is represented in assertion 2). The simply connected Lie group $G_{5.6}$ of $\mathbf{g}_{5.6}$ is isomorphic to the linear group of matrices

$$g(q, x, y, z, w) = \begin{pmatrix} 1 & 2w & w^2 - z & y - zw + \frac{w^3}{3} & q \\ 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $q, x, y, z, w \in \mathbb{R}$. For $i = 4, 5, 6$ the centre of $G_{5.i}$ is $Z = \exp(\langle e_1 \rangle)$. If $G_{5.i}$, $i = 4, 5, 6$, is the multiplication group $Mult(L)$ of L, then the centre $Z(L) = Z(e)$ of L is isomorphic to $\mathbb{R}$ (cf. Lemma 2.5 a).

The factor algebra $\mathbf{g}_{5.i}/\langle e_1\rangle$ for $i=4,5$ is different from the Lie algebra of $\mathcal{F}_4$. Therefore the factor loop $L/Z(e)$ is isomorphic to $\mathbb{R}^2$. According to Proposition 2.7 a) the Lie algebra $\mathbf{inn(L)}$ of the inner mapping group $Inn(L)$ of L is a 2-dimensional abelian subalgebra of $\mathbf{g}_{5.i}$, $i=4,5$, which does not contain any non-trivial ideal of $\mathbf{g}_{5.i}$.

Up to automorphisms of the Lie algebra $\mathbf{g}_{5.4}$ we get that $\mathbf{inn(L)}=\langle e_2,e_3\rangle$. Hence one has $Inn(L)=\{g(0,z,w,0,0), z,w\in\mathbb{R}\}$. The set

$$A=\{g(q,x+y,x-2y,x,y), q,x,y\in\mathbb{R}\}$$

is $Inn(L)$-connected left transversal in $G_{5.4}$ such that A generates $G_{5.4}$. Therefore Lemma 2.1 yields assertion 1).

By Proposition 2.7 a) there is an abelian normal subgroup P of the group $G_{5.5}$ such that the factor group $G_{5.5}/P\cong\mathbb{R}^2$. The Lie algebra $\mathbf{p}$ of the group P has one of the following forms: $\mathbf{p}_1=\langle e_1,e_2,e_3+k_1e_4\rangle$, $k_1\in\mathbb{R}$, $\mathbf{p}_2=\langle e_1,e_2,e_4\rangle$. As $\langle e_1\rangle$ is the centre of $\mathbf{g}_{5.5}$ we may choose $\mathbf{inn(L)}$ in the following way $\mathbf{inn(L)}_{1,l_1,l_2,k_1}=\langle e_2+l_1e_1,e_3+k_1e_4+l_2e_1\rangle$, $l_1,l_2,k_1\in\mathbb{R}$, $\mathbf{inn(L)}_{2,n_1,n_2}=\langle e_2+n_1e_1,e_4+n_2e_1\rangle$, $n_1,n_2\in\mathbb{R}$. The automorphism $\alpha(e_1)=e_1$, $\alpha(e_2)=e_2-l_1e_1$, $\alpha(e_3)=e_3-k_1e_4-l_2e_1$, $\alpha(e_4)=e_4$, $\alpha(e_5)=e_5$ of $\mathbf{g}_{5.5}$ maps $\mathbf{inn(L)}_{1,l_1,l_2,k_1}$ onto $\mathbf{inn(L)}_1=\langle e_2,e_3\rangle$ and the automorphism $\beta(e_1)=e_1$, $\beta(e_2)=e_2-n_1e_1$, $\beta(e_3)=e_3-n_1e_2$, $\beta(e_4)=e_4-n_2e_1$, $\beta(e_5)=e_5$ changes $\mathbf{inn(L)}_{2,n_1,n_2}$ onto $\mathbf{inn(L)}_2=\langle e_2,e_4\rangle$. The corresponding Lie groups are $Inn(L)_1=\{g(0,y,q,0,0), y,q\in\mathbb{R}\}$, $Inn(L)_2=\{g(0,y,0,z,0), y,z\in\mathbb{R}\}$. The set $A=\{g(x,zw+\frac{w^3}{2},z+\frac{w^2}{2},z,w), x,z,w\in\mathbb{R}\}$ is $Inn(L)_1$- and the set $B=\{g(x,qw,q,-\frac{w^2}{2},w), x,q,w\in\mathbb{R}\}$ is $Inn(L)_2$-connected left transversals in $G_{5.5}$. The set A and also the set B generates $G_{5.5}$. By Lemma 2.1 assertion 2) follows.

Finally, we prove that the Lie algebra $\mathbf{g}_{5.6}$ is not the Lie algebra of the multiplication group of a 3-dimensional connected simply connected topological proper loop L. If $\mathbf{g}_{5.6}$ is the Lie algebra of the multiplication group of L, then the Lie algebra $\mathbf{inn(L)}$ of the inner mapping group $Inn(L)$ of L is a 2-dimensional subalgebra of $\mathbf{g}_{5.6}$ which does not contain any non-trivial ideal of $\mathbf{g}_{5.6}$. Any 2-dimensional subalgebra of a nilpotent Lie algebra is abelian. Hence up to automorphisms of $\mathbf{g}_{5.6}$ the following subalgebras can occur as the Lie algebra $\mathbf{inn(L)}$: $\mathbf{inn_1(L)}=\langle e_2,e_3\rangle$, $\mathbf{inn_2(L)}=\langle e_2,e_4\rangle$. The corresponding Lie groups are $Inn_1(L)=\{g(0,x,y,0,0), x,y\in\mathbb{R}\}$ and $Inn_2(L)=\{g(0,x,0,z,0), x,z\in\mathbb{R}\}$. Arbitrary left transversals to the group $Inn_1(L)$ of $G_{5.6}$ are

$$A=\{g(q,f_1(q,z,w),f_2(q,z,w),z,w),\ q,z,w\in\mathbb{R}\},$$

$$B=\{g(m,h_1(m,n,p),h_2(m,n,p),n,p),\ m,n,p\in\mathbb{R}\},$$

respectively those to the group $Inn_2(L)$ are

$$C=\{g(q,f_3(q,y,w),y,f_4(q,y,w),w),\ q,y,w\in\mathbb{R}\},$$

$$D=\{g(m,h_3(m,n,p),n,h_4(m,n,p),p),\ m,n,p\in\mathbb{R}\},$$

where $f_i(q,z,w),h_i(m,n,p),f_j(q,y,w),h_j(m,n,p):\mathbb{R}^3\to\mathbb{R}$, $i=1,2$, $j=3,4$, are continuous functions with $f_i(0,0,0)=f_j(0,0,0)=0=h_i(0,0,0)=h_j(0,0,0)$.

The products $a^{-1}b^{-1}ab$ with $a \in A$, $b \in B$ are contained in $Inn_1(L)$ if the equation $wn - pz = 0$ holds for all $w, n, p, z \in \mathbb{R}$, which gives a contradiction.
The products $c^{-1}d^{-1}cd$ with $c = g(0, f_3(0,0,w), 0, f_4(0,0,w), w) \in C$, $d = g(0, h_3(0,n,p), n, h_4(0,n,p), p) \in D$ are contained in $Inn_2(L)$ if and only if the equations

$$pf_4(0,0,w) = wh_4(0,n,p) \tag{2}$$

$$2wh_3(0,n,p) - 2pf_3(0,0,w) = n(w^2 + 2wp) + pf_4(0,0,w)^2 - wh_4(0,n,p)^2 + f_4(0,0,w)(w^2p + 2n + \frac{1}{3}p^3 + wp^2) - h_4(0,n,p)(w^2p + wp^2 + \frac{1}{3}w^3) \tag{3}$$

are satisfied for all $p, w, n \in \mathbb{R}$. From equation (2) we obtain $f_4(0,0,w) = c_1 w$ and $h_4(0,n,p) = c_1 p$, where $c_1 \in \mathbb{R}$ is a constant. Putting these into equation (3) it reduces to

$$2wh_3(0,n,p) - 2pf_3(0,0,w) = \frac{2}{3}c_1 w^3 p + w^2(c_1^2 p + n) + w(2c_1 n + 2np - \frac{2}{3}c_1 p^3 - c_1^2 p^2). \tag{4}$$

Since the right hand side of (4) contains the term $w^2 n$ there are no functions $f_3(0,0,w)$ and $h_3(0,n,p)$ such that equation (4) holds and the claim is proved. ■

4. Six-dimensional nilpotent Lie groups which are multiplication groups of three-dimensional loops

Let L be a connected topological loop homeomorphic to $\mathbb{R}^3$ and having a 6-dimensional indecomposable nilpotent Lie group as the group $Mult(L)$ of L. If L is centrally nilpotent of class 2, then one has $Z(L) \cong \mathbb{R}$ and $L/Z(L) \cong \mathbb{R}^2$ (cf. Proposition 2.7 a). Moreover, the Lie algebra $\mathbf{g}$ of $Mult(L)$ has an abelian ideal $\mathbf{p}$ such that the factor algebra $\mathbf{g}/\mathbf{p}$ is isomorphic to $\mathbb{R}^2$. The Lie algebra $\mathbf{g}$ has a 3-dimensional abelian subalgebra $\mathbf{k}$ which does not contain any non-zero ideal of $\mathbf{g}$ and the normalizer $N_{\mathbf{g}}(\mathbf{k})$ of $\mathbf{k}$ in $\mathbf{g}$ is $\mathbf{p}$ (cf. Proposition 2.7 a) and Lemmata 2.1, 2.2).

Proposition 4.1. *Let $\mathbf{g}$ be a 6-dimensional indecomposable nilpotent Lie algebra with 1-dimensional centre. Let $\mathbf{p}$ be an abelian ideal of $\mathbf{g}$ such that the factor algebra $\mathbf{g}/\mathbf{p}$ is isomorphic to $\mathbb{R}^2$. Let $\mathbf{k}$ be a 3-dimensional abelian subalgebra of $\mathbf{g}$ which does not contain any non-zero ideal of $\mathbf{g}$ and such that the normalizer $N_{\mathbf{g}}(\mathbf{k})$ of $\mathbf{k}$ in $\mathbf{g}$ is $\mathbf{p}$. Then for the Lie algebras $\mathbf{g}$, $\mathbf{p}$ and for the Lie algebra $\mathbf{k}$, up to automorphisms of $\mathbf{g}$, we have one of the following cases:*
(a) $\mathbf{g}_1$: $[e_1,e_2] = e_3$, $[e_1,e_3] = e_6$, $[e_4,e_5] = e_6$ (cf. Case $L_{6,10}$ in [10]). The ideals $\mathbf{p}$ have one of the following forms: $\mathbf{p}_{1,1} = \langle e_2, e_3, e_5, e_6 \rangle$, $\mathbf{p}_{1,2} = \langle e_2, e_3, e_4 + ke_5, e_6 \rangle$, $k \in \mathbb{R}$ and $\mathbf{k}_1 = \langle e_2, e_3, e_5 \rangle$.
(b) $\mathbf{g}_2$: $[e_1,e_2] = e_3$, $[e_1,e_3] = e_4$, $[e_1,e_4] = e_6$, $[e_2,e_3] = e_6$, $[e_2,e_5] = e_6$, (cf. Case $L_{6,11}$ in [10]), $\mathbf{p}_2 = \langle e_3, e_4, e_5, e_6 \rangle$ and $\mathbf{k}_2 = \langle e_3, e_4, e_5 \rangle$.
(c) $\mathbf{g}_3$: $[e_1,e_2] = e_3$, $[e_1,e_3] = e_4$, $[e_1,e_4] = e_6$, $[e_2,e_5] = e_6$ (cf. Case $L_{6,12}$ in [10]). The ideals $\mathbf{p}_3$ have one of the following forms: $\mathbf{p}_{3,1} = \langle e_3, e_4, e_5, e_6 \rangle$, $\mathbf{p}_{3,2} = \langle e_2 + ke_5, e_3, e_4, e_6 \rangle$, $k \in \mathbb{R}$. The Lie algebras $\mathbf{k}$ have one of the following forms: $\mathbf{k}_{3,1} = \langle e_3, e_4, e_5 \rangle$, $\mathbf{k}_{3,2} = \langle e_2 + ke_5, e_3, e_4 \rangle$, $k \in \mathbb{R}$.

(d) $\mathbf{g}_4$: $[e_1,e_2]=e_3$, $[e_1,e_3]=e_5$, $[e_1,e_5]=e_6$, $[e_2,e_4]=e_5$, $[e_3,e_4]=e_6$, *(cf. Case* $L_{6,13}$ *in [10]),* $\mathbf{p}_4=\langle e_2,e_3,e_5,e_6\rangle$ *and* $\mathbf{k}_4=\langle e_2,e_3,e_5\rangle$.
(e) $\mathbf{g}_5$: $[e_1,e_2]=e_4$, $[e_1,e_3]=e_5$, $[e_2,e_4]=e_6$, $[e_3,e_5]=\varepsilon e_6$, *(cf. Case* $L_{6,19}$ *with* $\varepsilon=\pm 1$ *in [10]),* $\mathbf{p}_5=\langle e_1,e_4,e_5,e_6\rangle$ *and* $\mathbf{k}_5=\langle e_1,e_4,e_5\rangle$.
(f) $\mathbf{g}_6$: $[e_1,e_2]=e_4$, $[e_1,e_3]=e_5$, $[e_1,e_5]=e_6$, $[e_2,e_4]=e_6$, *(cf. Case* $L_{6,20}$ *in [10]),* $\mathbf{p}_6=\langle e_3,e_4,e_5,e_6\rangle$ *and* $\mathbf{k}_6=\langle e_3,e_4,e_5\rangle$.
(g) $\mathbf{g}_7$: $[e_1,e_2]=e_3$, $[e_1,e_3]=e_4$, $[e_1,e_4]=e_5$, $[e_1,e_5]=e_6$, $[e_2,e_3]=e_5$, $[e_2,e_4]=e_6$, *(cf. Case* $L_{6,15}$ *in [10]),* $\mathbf{p}_7=\langle e_3,e_4,e_5,e_6\rangle$ *and* $\mathbf{k}_7=\langle e_3,e_4,e_5\rangle$.
(h) $\mathbf{g}_8$: $[e_1,e_2]=e_3$, $[e_1,e_3]=e_4$, $[e_1,e_4]=e_5$, $[e_1,e_5]=e_6$, $[e_2,e_3]=e_6$, *(cf. Case* $L_{6,17}$ *in [10]),* $\mathbf{p}_8=\langle e_3,e_4,e_5,e_6\rangle$ *and* $\mathbf{k}_8=\langle e_3,e_4,e_5\rangle$.
(i) $\mathbf{g}_9$: $[e_1,e_2]=e_3$, $[e_1,e_3]=e_4$, $[e_1,e_4]=e_6$, $[e_2,e_3]=e_5$, $[e_2,e_5]=\pm e_6$, *(cf. Case* $L_{6,21}$ *with* $\varepsilon=\pm 1$ *in [10]),* $\mathbf{p}_9=\langle e_3,e_4,e_5,e_6\rangle$ *and* $\mathbf{k}_9=\langle e_3,e_4,e_5\rangle$.

Proof. The list of the 6-dimensional indecomposable nilpotent Lie algebras $\mathbf{g}$ in [10], pp. 646-647, gives that the triples $(\mathbf{g},\mathbf{p},\mathbf{k})$ of the Lie algebras have the forms as in the assertion. ∎

Proposition 4.2. *There does not exist* 3*-dimensional connected topological proper loop* L *such that* L *is centrally nilpotent of class* 2 *and the Lie algebra* $\mathbf{g}$ *of the multiplication group of* L *is one of the Lie algebras* $\mathbf{g}_i$, $i=7,8,9$, *of Proposition 4.1.*

Proof. By Lemma 2.4 we may assume that L is homeomorphic to $\mathbb{R}^3$. Linear representations of the simply connected Lie groups G_i of $\mathbf{g}_i$, $i=7,8,9$, are: For $i=7$ the group G_7 consists of matrices $g(w,z,y,x,q,p)=$

$$\begin{pmatrix} 1 & w & z+w^2 & zw-2y+\frac{w^3}{3} & 3x-2yw+\frac{w^2z}{2}+\frac{w^4}{12} & p \\ 0 & 1 & 2w & w^2-z & y-zw+\frac{w^3}{3} & q \\ 0 & 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (5)$$

for $i=8$ the group G_8 consists of matrices

$$g(w,z,y,x,q,p)=\begin{pmatrix} 1 & w & \frac{w^2}{2} & z+\frac{w^3}{6} & \frac{w^4}{24}-y+zw & p \\ 0 & 1 & w & \frac{w^2}{2} & \frac{w^3}{6} & q \\ 0 & 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

for $i=9$ the group G_9 consists of matrices

$$g(w,z,y,x,q,p)=\begin{pmatrix} 1 & z & -aw & \frac{-aw^2-z^2}{2} & \frac{-q}{4}-\frac{aw^3}{6}+\frac{yz-wz^2}{2} & p \\ 0 & 1 & 0 & -z & y-zw & q \\ 0 & 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 0 & 1 & w & 2y \\ 0 & 0 & 0 & 0 & 1 & 2z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

with $a=\pm 1$, $w,z,y,x,q,p\in\mathbb{R}$ (cf. Cases 6.20, 6.19, 6.18 in [8], pp. 16-18). The subgroup of the group G_i, $i=7,8,9$, which can occur as the inner mapping group $Inn(L)_i$ of L is the Lie group of the Lie algebra $\mathbf{k}_i$ given in Proposition 4.1 (g), (h), (i). Hence in all three cases we have $Inn(L)=\{g(0,0,y,x,q,0), x,y,q\in\mathbb{R}\}$. Arbitrary left transversals to the group $Inn(L)$ of the groups G_i, $i=7,8,9$, are:

$$A=\{g(w,z,f_1(w,z,p),f_2(w,z,p),f_3(w,z,p),p),\ w,z,p\in\mathbb{R}\},$$

$$B=\{g(m,n,g_1(m,n,u),g_2(m,n,u),g_3(m,n,u),u),\ m,n,u\in\mathbb{R}\},$$

where $f_i(w,z,p):\mathbb{R}^3\to\mathbb{R}$, $g_i(m,n,u):\mathbb{R}^3\to\mathbb{R}$, $i=1,2,3$, are continuous functions with $f_i(0,0,0)=0=g_i(0,0,0)$. We prove that none of the groups G_i, $i=7,8,9$, satisfies the condition that for all $a\in A$ and $b\in B$ one has $a^{-1}b^{-1}ab\in Inn(L)$. Therefore, the groups G_i, $i=7,8,9$, are not multiplication groups of L (cf. Lemma 2.1).
With the elements

$$a=g(w,z,f_1(w,z,0),f_2(w,z,0),f_3(w,z,0),0)\in A,$$

$$b=g(m,0,g_1(m,0,0),g_2(m,0,0),g_3(m,0,0),0)\in B$$

of G_7 the products $a^{-1}b^{-1}ab$ are contained in $Inn(L)$ if and only if the equation

$$-f_1(w,z,0)[mz+wm^2+mw^2+\frac{m^3}{3}]+f_2(w,z,0)[2wm+m^2]-mf_3(w,z,0)+$$

$$g_1(m,0,0)[m^2w+w^2m+2zm+2zw+\frac{w^3}{3}]-g_2(m,0,0)[2wm+w^2+2z]+$$

$$+wg_3(m,0,0)=\frac{m^2z^2-zm^2w^2}{2}-\frac{zwm^3+zmw^3}{3}-\frac{zm^4}{12}-mwz^2 \qquad (6)$$

is satisfied for all $w,z,m\in\mathbb{R}$. Equality (6) holds if and only if one has $f_3(w,z,0)=-(z+w^2)f_1(w,z,0)+2wf_2(w,z,0)+z^2w+\frac{1}{3}zw^3$ and $g_3(m,0,0)=-m^2g_1(m,0,0)+2mg_2(m,0,0)$. Putting this into equation (6) we obtain

$$-f_1(w,z,0)[wm^2+\frac{m^3}{3}]+m^2f_2(w,z,0)+g_1(m,0,0)[w^2m+2zm+2zw+\frac{w^3}{3}]-$$

$$-g_2(m,0,0)[w^2+2z]=\frac{m^2z^2-zm^2w^2}{2}-\frac{zwm^3}{3}-\frac{zm^4}{12}. \qquad (7)$$

Substituting $f_2(w,z,0) = wf_1(w,z,0) + w^2 + 2z + \frac{z^2}{2} - \frac{zw^2}{2} + h_2(w,z)$, $g_2(m,0,0) = mg_1(m,0,0) + m^2 + j_2(m)$ into (7) with the continuous functions $h_2 : \mathbb{R}^2 \to \mathbb{R}$, $j_2 : \mathbb{R} \to \mathbb{R}$, $h_2(0,0) = 0 = j_2(0)$, we have

$$\frac{m^3}{3} f_1(w,z,0) - g_1(m,0,0)[2zw + \frac{w^3}{3}] - m^2 h_2(w,z) + j_2(m)[w^2 + 2z] = \frac{zwm^3}{3} + \frac{zm^4}{12}. \tag{8}$$

Since on the right hand side of (8) there is the term $-\frac{zm^4}{12}$ there does not exist functions $f_1(w,z,0)$, $h_2(w,z)$, $g_1(m,0,0)$, $j_2(m)$ such that for all w, z, $m \in \mathbb{R}$ equation (8) holds. This contradiction excludes the group G_7.

Taking the elements

$$c = g(0,z,f_1(0,z,0), f_2(0,z,0), f_3(0,z,0), 0) \in A,$$

$$d = g(m,n,g_1(m,n,0), g_2(m,n,0), g_3(m,n,0), 0) \in B$$

of G_8 the products $c^{-1}d^{-1}cd$ are contained in $Inn(L)$ if and only if the equation

$$nf_1(0,z,0) - zg_1(m,n,0) = \frac{zm^4}{48} - \frac{m^3 f_1(0,z,0)}{12} + \frac{m^2 f_2(0,z,0)}{4} + \frac{m(z^2 - f_3(0,z,0))}{2} \tag{9}$$

is satisfied for all $z, m, n \in \mathbb{R}$. As the right hand side of (9) does not depend on the variable n so is the left hand side. Therefore one has $f_1(0,z,0) = cz$, $g_1(m,n,0) = cn$, $c \in \mathbb{R}$. But there are no functions $f_2(0,z,0)$, $f_3(0,z,0)$ such that the equation

$$0 = \frac{zm^4}{48} - \frac{czm^3}{12} + \frac{m^2 f_2(0,z,0)}{4} + \frac{m(z^2 - f_3(0,z,0))}{2}$$

holds for all $z, m \in \mathbb{R}$. This contradiction excludes the group G_8.

The products $e^{-1}f^{-1}ef$ of the elements

$$e = g(w,z,f_1(w,z,0), f_2(w,z,0), f_3(w,z,0), 0) \in A,$$

$$f = g(m,n,g_1(m,n,0), g_2(m,n,0), g_3(m,n,0), 0) \in B$$

of G_9 are contained in $Inn(L)$ if and only if with $a = \pm 1$ the equation

$$f_1(w,z,0)(2wm + 3an^2 + 3azn + m^2) - f_2(w,z,0)m + \frac{3an}{2} f_3(w,z,0) -$$

$$-g_1(m,n,0)(2wm + 3anz + 3az^2 + w^2) + g_2(m,n,0)w - \frac{3az}{2} g_3(m,n,0) =$$

$$(z-n)(w^2 m + wm^2) - awn^3 - amz^3 + \frac{zm^3 - nw^3}{3} \tag{10}$$

is satisfied for all $w, z, m, n \in \mathbb{R}$. Equation (10) holds if and only if one has $f_2(w,z,0) = 2wf_1(w,z,0) - zw^2 - az^3$, $f_3(w,z,0) = -2zf_1(w,z,0) - \frac{2}{9}aw^3$, $g_2(m,n,0) = 2mg_1(m,n,0) - m^2 n - an^3$, $g_3(m,n,0) = -2ng_1(m,n,0) - \frac{2}{9}am^3$. Putting this into equation (10) we obtain

$$f_1(w,z,0)(3an^2 + m^2) - g_1(m,n,0)(3az^2 + w^2) = zwm^2 - w^2 nm. \tag{11}$$

As on the right hand side of (11) there is no n^2 and z^2 we have $f_1(w,z,0) = b(3az^2+w^2)$, $g_1(m,n,0) = b(3an^2+m^2)$, where $b \in \mathbb{R}$ is an arbitrary constant. Substituting this into (11) we get the contradiction that for all $z,w,n,m \in \mathbb{R}$ one has $0 = zwm^2 - nmw^2$. This excludes the group G_9. ■

Theorem 4.3. *Let L be a connected simply connected topological proper loop of dimension 3 such that its multiplication group $Mult(L)$ is a 6-dimensional indecomposable nilpotent Lie group. If L is centrally nilpotent of class 2, then the following Lie groups are the multiplication groups $Mult(L)$ and the following subgroups are the inner mapping groups $Inn(L)$ of L:*
1) $Mult(L)_1$ is the direct product $\mathcal{F}_4 \times_Z \mathcal{F}_3$ with amalgamated centre Z the multiplication of which is given by $g(u_1,y_1,x_1,w_1,z_1,q_1)g(u_2,y_2,x_2,w_2,z_2,q_2) =$

$$g(u_1+u_2, y_1+y_2, x_1+x_2+u_1y_2, w_1+w_2, z_1+z_2, q_1+q_2+u_1x_2+\frac{u_1^2y_2}{2}+w_1z_2).$$

$Inn(L)_1$ is the subgroup $\{g(0,y,x,0,z,0), x,y,z \in \mathbb{R}\}$.
2) The multiplication of $Mult(L)_2$ is given by

$$g(w_1,z_1,y_1,x_1,p_1,q_1)g(w_2,z_2,y_2,x_2,p_2,q_2) = g(w_1+w_2, z_1+z_2, y_1+y_2+w_1z_2,$$

$$x_1+x_2+w_1y_2+\frac{w_1^2z_2}{2}, p_1+p_2, q_1+q_2+2w_1x_2+(w_1^2-z_1)y_2+(p_1+y_1-z_1w_1+\frac{w_1^3}{3})z_2).$$

$Inn(L)_2$ is the subgroup $\{g(0,0,y,x,p,0), x,y,p \in \mathbb{R}\}$.
3) The multiplication of $Mult(L)_3$ is given by

$$g(w_1,z_1,y_1,x_1,p_1,q_1)g(w_2,z_2,y_2,x_2,p_2,q_2) = g(w_1+w_2, z_1+z_2, y_1+y_2+w_1z_2,$$

$$x_1+x_2+w_1y_2+\frac{w_1^2z_2}{2}, p_1+p_2, q_1+q_2+w_1x_2+\frac{w_1^2y_2}{2}+(p_1+\frac{w_1^3}{6})z_2).$$

$Inn(L)_3$ is one of the subgroups $Inn(L)_{3,1} = \{g(0,0,y,x,p,0), x,y,p \in \mathbb{R}\}$, $Inn(L)_{3,2} = \{g(0,z,y,x,kz,0), x,y,z \in \mathbb{R}\}$, $k \in \mathbb{R}$.
4) The multiplication of $Mult(L)_4$ is given by

$$g(w_1,z_1,y_1,x_1,p_1,q_1)g(w_2,z_2,y_2,x_2,p_2,q_2) = g(w_1+w_2, z_1+z_2, y_1+y_2+w_1z_2,$$

$$x_1+x_2+w_1y_2+(p_1+\frac{w_1^2}{2})z_2, p_1+p_2, q_1+q_2+y_2(p_1+\frac{w_1^2}{2})+w_1(x_2+z_2p_1)+z_2\frac{w_1^3}{6}).$$

5) The multiplication of $Mult(L)_5$ is given by

$$g(z_1,y_1,w_1,x_1,q_1,p_1)g(z_2,y_2,w_2,x_2,q_2,p_2) = g(z_1+z_2, y_1+y_2, w_1+w_2,$$

$$x_1+x_2+w_1z_2, q_1+q_2+w_1y_2, p_1+p_2+y_1q_2+z_1x_2+(y_1w_1-q_1)y_2+(z_1w_1-x_1)z_2).$$

6) The multiplication of $Mult(L)_6$ is given by

$$g(x_1,y_1,w_1,z_1,q_1,p_1)g(x_2,y_2,w_2,z_2,q_2,p_2) = g(x_1+x_2, y_1+y_2, w_1+w_2,$$

$$z_1+z_2+\frac{x_1w_2-x_2w_1-y_1w_2-y_2w_1}{2}, q_1+q_2-\frac{x_1w_2+x_2w_1-y_1w_2+y_2w_1}{2},$$
$$p_1+p_2+(x_1-y_1)(z_2+q_2)-w_1w_2-(z_1-q_1)(x_2+y_2)).$$

7) The multiplication of $Mult(L)_7$ is given by

$$g(w_1,p_1,z_1,y_1,q_1,x_1)g(w_2,p_2,z_2,y_2,q_2,x_2)=g(w_1+w_2,p_1+p_2,z_1+z_2,$$
$$y_1+y_2+w_1z_2+p_1p_2, q_1+q_2+w_1p_2, x_1+x_2+w_1y_2+p_2q_1+\frac{w_1^2z_2}{2}).$$

$Inn(L)_i$, $i=4,5,6,7$, is the subgroup $\{g(0,0,z,y,q,0), z,y,q\in\mathbb{R}\}$.
The multiplication groups $Mult(L)_i$, $i=2,\cdots,7$ are precisely the 6-dimensional simply connected indecomposable nilpotent Lie groups with 1-dimensional centre and 3-dimensional commutator subgroup.

Proof. As $\dim(Mult(L))=6$ by Proposition 2.9 we get that $Mult(L)$ has 1-dimensional centre and hence the simply connected loop L has also 1-dimensional centre $Z(L)\cong\mathbb{R}$ (cf. Theorem 11 in [1]). Since L is centrally nilpotent of class 2 it follows from Propositions 4.1, 4.2 that the pairs $(\mathbf{g}_i,\mathbf{k}_i)$, $i=1,\cdots,6$, in Proposition 4.1 can be occur as the Lie algebras of the group $Mult(L)$ and the subgroup $Inn(L)$ of L. According to the list of the 6-dimensional indecomposable nilpotent Lie algebras $\mathbf{g}$ in [10], pp. 646-647, precisely the Lie algebras $\mathbf{g}_i$, $i=2,\cdots,6$, in Proposition 4.1 have 1-dimensional centre and 3-dimensional commutator ideal. Linear representations of the simply connected Lie groups of $\mathbf{g}_i$ are given in this order by the direct product $Mult(L)_1=\mathcal{F}_4\times_Z\mathcal{F}_3$, by the matrix groups $Mult(L)_2=G_{6,15}$ in [8] (p. 15), $Mult(L)_3=G_{6,17}$ in [8] (p. 16), $Mult(L)_4=G_{6,16}$ in [8] (p. 16), $Mult(L)_5=G_{6,14}$ with $a=1$ in [8] (p. 15), $Mult(L)_6=G_{6,14}$ with $a=-1$ in [8] (p. 15), $Mult(L)_7=G_{6,13}$ in [8] (p. 14). Using these linear representations the Lie groups of the Lie algebras $\mathbf{k}_i$ are the Lie groups $Inn(L)_i$, $i=1,\cdots,7$, of Theorem 4.3. The set

$$A_1=\{g(u,u^2,u,w,w,q), u,w,q\in\mathbb{R}\},\ \text{respectively}$$

$$A_2=\{g(w,z,wz,-\frac{1}{2}z^2+\frac{1}{2}w^2z,-\frac{w^3}{3}-2wz,q), w,z,q\in\mathbb{R}\}$$

is $Inn(L)_1$-, respectively $Inn(L)_2$-connected left transversal in $Mult(L)_1$, respectively in $Mult(L)_2$. The set

$$A_{3,1}=\{g(w,z,wz,\frac{w^2z}{2},-\frac{w^3}{6},q), w,z,q\in\mathbb{R}\}, \text{respectively}$$

$$A_{3,2}=\{g(w,p+\frac{w^3}{3},wp+\frac{w^4}{6},\frac{w^2p}{2}+\frac{w^5}{12},p,q), p,q,w\in\mathbb{R}\}$$

is $Inn(L)_{3,1}$-, respectively $Inn(L)_{3,2}$-connected left transversal in $Mult(L)_3$. The set

$$A_4=\{g(w,z,zw-\frac{w^3}{6},zw^2-\frac{w^4}{6},z^2w+\frac{zw^3}{3}-\frac{w^5}{12},q),\ w,z,q\in\mathbb{R}\},$$

$$A_5 = \{g(z, y, y^2 - z^2, \frac{zy^2 - yz^2 - z^3 - y^3}{2}, \frac{z^3 + y^3 - zy^2 - yz^2}{2}, p),\ z, y, p \in \mathbb{R}\},$$

$$A_6 = \{g(z, y, y^2 + z^2, z, y, p),\ z, y, p \in \mathbb{R}\},\ A_7 = \{g(w, p, w^2, w^3, 0, x),\ w, p, x \in \mathbb{R}\}$$

is in this order $Inn(L)_i$-connected left transversal in $Mult(L)_i$, $i = 4, 5, 6, 7$. Moreover, for all $i = 1, \cdots, 7$, the set A_i generates the group $Mult(L)_i$. Hence Lemma 2.1 is satisfied for all these transversals and the theorem is proved. ■

5. 3-dimensional loops L with 6-dimensional indecomposable nilpotent multiplication groups have nilpotency class 2

Proposition 5.1. *Let $\mathbf{g}$ be a 6-dimensional indecomposable nilpotent Lie algebra with 1-dimensional centre $\mathbf{z}$. Let $\mathbf{s}$ be an ideal of $\mathbf{g}$ such that $\mathbf{z} \le \mathbf{s}$ and the factor algebra $\mathbf{g}/\mathbf{s}$ is an elementary filiform Lie algebra $\mathbf{f}_m$ with $m \ge 4$. Let $\mathbf{v}$ be a 5-dimensional ideal of $\mathbf{g}$ such that $\mathbf{g}' < \mathbf{v}$ and $\mathbf{s} < \mathbf{v}$. Let $\mathbf{k}$ be a 3-dimensional subalgebra containing no ideal $\neq 0$ of $\mathbf{g}$ with the properties $\mathbf{k} < \mathbf{v}$, $dim(\mathbf{k} \cap \mathbf{s}) = 1$ if $\mathbf{z} \neq \mathbf{s}$, otherwise $\mathbf{k} \cap \mathbf{s} = 0$, $\mathbf{v}$ has an ideal $\mathbf{n}$ with $\mathbf{n} < \mathbf{k}$, $\mathbf{v}/\mathbf{n} \cong \mathbf{f}_m$, $m \ge 3$.*
I) There does not exist any subalgebra $\mathbf{i} \cong \mathbf{f}_m$, $m \ge 4$ such that $\mathbf{z} < \mathbf{i} < \mathbf{v}$, $dim(\mathbf{k} \cap \mathbf{i}) \ge 2$, $\mathbf{i}$ complements $\mathbf{n}$ in $\mathbf{v}$ and the centre of $\mathbf{g}/\mathbf{s}$ is isomorphic to the factor algebra $\mathbf{i}/(\mathbf{i} \cap (\mathbf{k} \oplus \mathbf{z}))$.
II) Assume that $\mathbf{g}$ has a subalgebra $\mathbf{i} \cong \mathbb{R}^2$ such that $\mathbf{z} < \mathbf{i} < \mathbf{v}$, $\mathbf{k}$ complements $\mathbf{i}$ in $\mathbf{v}$ and the centre of $\mathbf{g}/\mathbf{s}$ is isomorphic to $\mathbf{i}/\mathbf{z}$. Then the Lie algebra $\mathbf{g}$ coincides with $\mathbf{g}_7$ in Proposition 4.1 and up to automorphisms of $\mathbf{g}$ the subalgebra $\mathbf{k}$ is $\mathbf{k} = \langle e_2, e_3, e_5 \rangle$.

Proof. Applying the list of [10], pp. 646-647, we get for the pairs $(\mathbf{g}, \mathbf{s})$ the following:
(a) $\mathbf{g}_2$ in Proposition 4.1 and $\mathbf{s} = \langle e_5 + ke_4, e_6 \rangle$, $k \in \mathbb{R}$.
(b) $\mathbf{g}_3$ in Proposition 4.1 and $\mathbf{s} = \langle e_5 + ke_4, e_6 \rangle$, $k \in \mathbb{R}$.
(c) $\mathbf{g}$: $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_1, e_4] = e_5$, $[e_2, e_3] = e_5$, $[e_2, e_5] = e_6$, $[e_3, e_4] = -e_6$, (cf. Case $L_{6,14}$ in [10]) and $\mathbf{s} = \langle e_5, e_6 \rangle$.
(d) $\mathbf{g}_7$ in Proposition 4.1 and $\mathbf{s} = \langle e_5, e_6 \rangle$.
(e) $\mathbf{g}$: $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_1, e_4] = e_5$, $[e_2, e_5] = e_6$, $[e_3, e_4] = -e_6$ (cf. Case $L_{6,16}$ in [10]) and $\mathbf{s} = \langle e_5, e_6 \rangle$.
(f) $\mathbf{g}_8$ in Proposition 4.1 and the ideal $\mathbf{s}$ is either $\mathbf{s}_{8,1} = \langle e_5, e_6 \rangle$ or $\mathbf{s}_{8,2} = \langle e_6 \rangle$.
(g) $\mathbf{g}_{9,\pm}$ in Proposition 4.1 and the ideal $\mathbf{s}$ is either $\mathbf{s}_{9,1} = \langle e_5, e_6 \rangle$ or $\mathbf{s}_{9,2} = \langle e_4, e_6 \rangle$.
In all cases (a) till (g) the centre $\mathbf{z}$ of the Lie algebra $\mathbf{g}$ is $\langle e_6 \rangle$. In cases (a) till (e) the centre $\mathbf{z}(\mathbf{g}/\mathbf{s})$ of the filiform factor algebra $\mathbf{g}/\mathbf{s}$ is isomorphic to $\langle e_4 \rangle$. For $i = \{8, 9\}$ the centre of $\mathbf{g_i}/\mathbf{s}_{i,1}$, respectively of $\mathbf{g_i}/\mathbf{s}_{i,2}$ is isomorphic to $\langle e_4 \rangle$, respectively to $\langle e_5 \rangle$. In all cases $\langle e_4 \rangle$ and $\langle e_5 \rangle$ are in the commutator ideal $\mathbf{g}'$ of $\mathbf{g}$. The subalgebra $\mathbf{k}$ does not contain any proper ideal of $\mathbf{g}$ and with the exception of $\mathbf{s}_{8,2}$ the intersection $\mathbf{k} \cap \mathbf{s}$ has dimension 1. Hence in cases (a) and (b) the element $e_5 + ke_4 + ae_6 \in \mathbf{s}$, in cases (c) till (f) the element $e_5 + ae_6 \in \mathbf{s}$, in case (g) either the element $e_5 + ae_6 \in \mathbf{s}_{9,1}$ or the element $e_4 + ae_6 \in \mathbf{s}_{9,2}$, $a, k \in \mathbb{R}$, is contained in $\mathbf{k}$.
In all cases (a) till (g) there are two ideals $\mathbf{v}$ of $\mathbf{g}$ with codimension 1 such that $\mathbf{g}' < \mathbf{v}$ and $\mathbf{s} < \mathbf{v}$: $\mathbf{v}_{1,l} = \langle e_1 + le_2, e_3, e_4, e_5, e_6 \rangle$, $l \in \mathbb{R}$, and $\mathbf{v}_2 = \langle e_2, e_3, e_4, e_5, e_6 \rangle$.

For the subalgebras $\mathbf{i}$ and $\mathbf{k}$ of $\mathbf{g}$ having the properties as in I) we obtain:
In cases (d) and (f) the Lie algebra $\mathbf{i}$ coincides with $\mathbf{v}_{1,l}$ and the Lie algebra $\mathbf{k}$ has the form $\mathbf{k}_{l_1,l_2,l_3} = \langle e_3 + l_1 e_6, e_4 + l_2 e_6, e_5 + l_3 e_6 \rangle$, $l_i \in \mathbb{R}$, $i = 1, 2, 3$. The factor algebra $\mathbf{v}_{1,l}/(\mathbf{k} \oplus \mathbf{z})$ is isomorphic to $\langle e_1 + l e_2 \rangle$, $l \in \mathbb{R}$. In cases (a), (b), (g) the Lie algebra $\mathbf{i} < \mathbf{v}_{1,l}$ is $\mathbf{i} = \langle e_1, e_3, e_4, e_6 \rangle$ with the Lie brackets $[e_1, e_3] = e_4$, $[e_1, e_4] = e_6$, $l = 0$, and the Lie algebra $\mathbf{k}$ coincides with $\mathbf{k}_{l_1,l_2,l_3}$. The factor algebra $\mathbf{i}/(\mathbf{i} \cap (\mathbf{k} \oplus \mathbf{z}))$ is isomorphic to $\langle e_1 \rangle$. In case (g) for the Lie algebra $\mathbf{i}$ we obtain also the possibility $\mathbf{i} = \langle e_2, e_3, e_5, e_6 \rangle < \mathbf{v}_2$ with the Lie brackets $[e_2, e_3] = e_5$, $[e_2, e_5] = \pm e_6$ and the subalgebra $\mathbf{k}$ is $\mathbf{k}_{l_1,l_2,l_3}$. The factor algebra $\mathbf{i}/(\mathbf{i} \cap (\mathbf{k} \oplus \mathbf{z}))$ is isomorphic to $\langle e_2 \rangle$. Since $\langle e_1 + l e_2 \rangle$, $l \in \mathbb{R}$, $\langle e_1 \rangle$, $\langle e_2 \rangle$ are not in the commutator ideal of $\mathbf{g}$ the factor algebra $\mathbf{i}/(\mathbf{i} \cap (\mathbf{k} \oplus \mathbf{z}))$ is not isomorphic to the centre of $\mathbf{g}/\mathbf{s}$. This proves assertion I) of the proposition.
Now we seek for the subalgebras $\mathbf{i}$ and $\mathbf{k}$ of $\mathbf{g}$ having the properties as in II).
In cases (a), (b), (g) the factor algebra $\mathbf{v}_{1,l}/\mathbf{n} = \mathbf{v}_{1,0}/\langle e_5 + l_3 e_6 \rangle$, $l_3 \in \mathbb{R}$ is isomorphic to $\langle e_1, e_3, e_4, e_6 \rangle \cong \mathbf{f}_4$. In cases (d) and (f) the factor algebra $\mathbf{v}_{1,l}/\mathbf{n}$ is $\mathbf{v}_{1,l}$. In these cases the Lie algebra $\mathbf{i}$ has the form $\mathbf{i}_1 = \langle e_6, e_1 \rangle$ and the Lie algebra $\mathbf{k}_1$ coincides with $\mathbf{k}_{l_1,l_2,l_3}$. The factor algebra $\mathbf{i}/\mathbf{z}$ is isomorphic to $\langle e_1 \rangle$.
In case (b) the factor algebra $\mathbf{v}_2/\mathbf{n} = \mathbf{v}_2/\langle e_3 + l_1 e_6, e_4 + l_2 e_6 \rangle$, $l_1, l_2 \in \mathbb{R}$ is isomorphic to $\langle e_2, e_5, e_6 \rangle \cong \mathbf{f}_3$. Hence for the subalgebra $\mathbf{i} < \mathbf{v}_2$ we get $\mathbf{i}_2 = \langle e_2, e_6 \rangle$ and $\mathbf{k}_2 = \mathbf{k}_{l_1,l_2,l_3}$. The factor algebra $\mathbf{i}/\mathbf{z}$ is isomorphic to $\langle e_2 \rangle$.
In cases (c) and (e) for the factor algebra $\mathbf{v}_{1,l}/\mathbf{n}$ we obtain $\mathbf{v}_{1,0}/\langle e_5 \rangle$ which is isomorphic to $\langle e_3, e_1, e_4, e_6 \rangle \cong \mathbf{f}_4$. The subalgebra $\mathbf{i} < \mathbf{v}_{1,0}$ has the form $\mathbf{i}_3 = \langle e_6, e_3 \rangle$ and $\mathbf{k}_3$ coincides with $\mathbf{k}_{n_1,n_2} = \langle e_1 + n_1 e_6, e_4 + n_2 e_6, e_5 \rangle$, $n_1, n_2 \in \mathbb{R}$. The factor algebra $\mathbf{i}/\mathbf{z}$ is isomorphic to $\langle e_3 \rangle$.
In case (d) the factor algebra $\mathbf{v}_2/\mathbf{n} = \mathbf{v}_2/\langle e_3 + a_2 e_6 + a_1 e_4, e_5 + a_1 e_6 \rangle$ for some $a_1, a_2 \in \mathbb{R}$ is isomorphic to $\langle e_2, e_4, e_6 \rangle \cong \mathbf{f}_3$. The subalgebras $\mathbf{i} < \mathbf{v}_2$ and $\mathbf{k} < \mathbf{v}_2$ have one of the following forms:
$\mathbf{i}_4 = \langle e_4, e_6 \rangle$, $\mathbf{k}_4 = \mathbf{k}_{a_1,a_2,a_3,a_4} = \langle e_5 + a_1 e_6, e_3 + a_2 e_6 + a_1 e_4, e_2 + a_3 e_6 + a_4 e_4 \rangle$, $a_i \in \mathbb{R}$;
$\mathbf{i}_5 = \mathbf{i}_3$ and $\mathbf{k}_5 = \mathbf{k}_{a_1,a_2,a_3,a_4}$ with $a_1 \neq 0$;
$\mathbf{i}_6 = \mathbf{i}_2$ and for the Lie algebra $\mathbf{k}$ we get the following possibilities: $\mathbf{k}_{6,1} = \mathbf{k}_{l_1,l_2,l_3}$, $\mathbf{k}_{6,2} = \langle e_5, e_4 + l_2 e_6 + a_2 e_2, e_3 + l_1 e_6 \rangle$, $a_2 \neq 0$.
In case (f) the factor algebra $\mathbf{v}_2/\mathbf{n} = \mathbf{v}_2/\langle e_4 + a_2 e_6, e_5 + a_1 e_6 \rangle$, $a_1, a_2 \in \mathbb{R}$ is isomorphic to $\langle e_2, e_3, e_6 \rangle \cong \mathbf{f}_3$. The subalgebras $\mathbf{i}$ and $\mathbf{k}$ of $\mathbf{v}_2$ have one of the following forms: $\mathbf{i}_7 = \mathbf{i}_3$ and $\mathbf{k}_7 = \langle e_5 + a_1 e_6, e_4 + a_2 e_6, e_2 + a_3 e_6 + b_3 e_3 \rangle$,
$\mathbf{i}_8 = \mathbf{i}_2$ and $\mathbf{k}_8 = \mathbf{k}_{l_1,l_2,l_3}$. In case (g) the factor algebra $\mathbf{v}_2/\mathbf{n} = \mathbf{v}_2/\langle e_4 + a_2 e_6 \rangle$ is isomorphic to $\langle e_2, e_3, e_5, e_6 \rangle \cong \mathbf{f}_4$. For the subalgebra $\mathbf{i}$ of $\mathbf{v}_2$ we get $\mathbf{i}_9 = \mathbf{i}_2$ and the Lie algebra $\mathbf{k}_9$ coincides with $\mathbf{k}_{l_1,l_2,l_3}$. For $\mathbf{i}_1$ and $\mathbf{i}_2$ the factor algebra $\mathbf{i}/\mathbf{z}$ is not isomorphic to $\mathbf{z}(\mathbf{g}/\mathbf{s})$ because $\langle e_1 \rangle$ and $\langle e_2 \rangle$ are not in the commutator ideal of $\mathbf{g}$. In cases (c) till (f) the factor algebra $\mathbf{i}_3/\mathbf{z}$ is not isomorphic to $\mathbf{z}(\mathbf{g}/\mathbf{s})$. In case (d) the factor algebras $\mathbf{i}_4/\mathbf{z}$, $\mathbf{z}(\mathbf{g}/\mathbf{s})$ are isomorphic to $\langle e_4 \rangle$.
Using the automorphism $\gamma(e_6) = e_6$, $\gamma(e_5) = e_5 - a_1 e_6$, $\gamma(e_4) = e_4 - a_1 e_5 + a_1^2 e_6$, $\gamma(e_3) = e_3 - a_1 e_4 + a_1^2 e_5 - (a_2 + a_1^3) e_6$, $\gamma(e_2) = e_2 - a_1 e_3 - a_4 e_4 + a_1 a_4 e_5 - (a_3 + a_4 a_1^2) e_6$, $\gamma(e_1) = e_1 - (a_4 + a_1^2) e_3 + (a_1 a_4 + a_2 + a_1^3) e_4$ of $\mathbf{g}_7$ the Lie algebra $\mathbf{k}_{a_1,a_2,a_3,a_4}$ is reduced to $\mathbf{k} = \langle e_2, e_3, e_5 \rangle$, which proves the assertion II). ■

Theorem 5.2. *There does not exist any 3-dimensional connected topological proper loop L such that L is centrally nilpotent of class 3 and the multiplication* *group $Mult(L)$ of L is a 6-dimensional indecomposable nilpotent Lie group.*

Proof. By Lemma 2.4 we may assume that L is homeomorphic to $\mathbb{R}^3$. As L is centrally nilpotent of class 3 by Proposition 2.7 b) the factor loop $L/Z(L)$ is isomorphic to an elementary filiform loop $L_{\mathcal{F}}$. Moreover, L has a 2-dimensional normal subloop M isomorphic either to a loop $L_{\mathcal{F}}$ or to the group $\mathbb{R}^2$. If $M \cong L_{\mathcal{F}}$, then the Lie algebra $\mathbf{mult(L)}$ of the multiplication group $Mult(L)$ of L and the Lie algebra $\mathbf{inn(L)}$ of the inner mapping group $Inn(L)$ of L satisfy the same conditions given in Proposition 5.1 I) for the Lie algebra $\mathbf{g}$ and the subalgebra $\mathbf{k}$. If $M \cong \mathbb{R}^2$, then $\mathbf{mult(L)}$ and $\mathbf{inn(L)}$ have the same properties as $\mathbf{g}$ and $\mathbf{k}$ in Proposition 5.1 II). Therefore the pair $(\mathbf{g}, \mathbf{k})$ in Proposition 5.1 II) can occur as the pair $(\mathbf{mult(L)}, \mathbf{inn(L)})$ of L. In Proposition 5.1 we have $\mathbf{i} = \mathbf{mult(M)}$.
The simply connected Lie group G of $\mathbf{g}$ is isomorphic to the matrix group (5) (cf. Case 6.20 in [8], p. 18) and the Lie group of $\mathbf{inn(L)} = \langle e_2, e_3, e_5 \rangle$ is $Inn(L) = \{g(0, z, y, 0, q, 0), z, y, q \in \mathbb{R}\}$. Arbitrary left transversals to the group $Inn(L)$ of the group G are:

$$A = \{g(w, f_1(w,x,p), f_2(w,x,p), x, f_3(w,x,p), p), w, x, p \in \mathbb{R}\},$$

$$B = \{g(m, g_1(m,n,r), g_2(m,n,r), n, g_3(m,n,r), r), m, n, r \in \mathbb{R}\},$$

where $f_i(w,x,p) : \mathbb{R}^3 \to \mathbb{R}$, $g_i(m,n,r) : \mathbb{R}^3 \to \mathbb{R}$, $i = 1,2,3$, are continuous functions with $f_i(0,0,0) = 0 = g_i(0,0,0)$. The set $\{a^{-1}b^{-1}ab; a \in A, b \in B\}$ is contained in $Inn(L)$ if and only if the following equations are satisfied:

$$0 = \frac{f_1(w,x,p)m^2 - g_1(m,n,r)w^2}{2} + (f_1(w,x,p) - g_1(m,n,r))wm + \\ + g_2(m,n,r)w - f_2(w,x,p)m \tag{12}$$

$$3f_1(w,x,p)(\frac{m^2}{2} + wm) - 3f_2(w,x,p)m = 3g_1(m,n,r)(\frac{w^2}{2} + wm) - 3g_2(m,n,r)w \tag{13}$$

$$r(f_1(w,x,p), f_2(w,x,p), g_1(m,n,r), g_2(m,n,r), m, w) = xm^2 - w^2n + 2xwm - \\ -2nwm + 2xg_1(m,n,r) - 2f_1(w,x,p)n + wg_3(m,n,r) - mf_3(w,x,p), \tag{14}$$

where $r(f_1(w,x,p), f_2(w,x,p), g_1(m,n,r), g_2(m,n,r), m, w)$ are the terms which depend on $f_1(w,x,p)$, $f_2(w,x,p)$, $g_1(m,n,r)$, $g_2(m,n,r)$, m, w. To satisfy equation (12) it is necessary that we have $f_2(w,x,p) = f_1(w,x,p)w$, $g_2(m,n,r) = g_1(m,n,r)m$. Putting these forms of the functions f_2, g_2 into (12) it is reduced to $0 = \frac{f_1(w,x,p)m^2 - g_1(m,n,r)w^2}{2}$. This yields that $f_1(w,x,p) = f_1(w) = cw^2$, $g_1(m,n,r) = g_1(m) = cm^2$, $c \in \mathbb{R}$. Therefore we get $f_2(w,x,p) = f_2(w) = cw^3$ and $g_2(m,n,r) = g_2(m) = cm^3$. Using these forms of the functions f_i, g_i, $i = 1,2$, both sides of equation (13) are the same. Putting these forms of the functions f_i, g_i, $i = 1,2$, into (14) we obtain that the left hand side of (14) depends only on the variables m, w. Hence the right hand side is independent of the variables n, x, p, r. This is the case precisely if one has $c = -\frac{1}{2}$ and $f_3(w,x,p) = 2xw$, $g_3(m,n,r) = 2nm$. Hence the right hand side of (14) is 0, but the left hand side is equal to $r(m,w) = \frac{1}{2}m^3w^3 + \frac{1}{3}(w^5m - wm^5) - \frac{1}{6}(2w^4m^2 + w^2m^4)$ which is different from 0. This contradiction proves the assertion. ■

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