

# Invariant metrizable and projective metrizable on Lie groups and homogeneous spaces

Ioan Bucataru, Tamás Milkovszki and Zoltán Muzsnay

## Abstract

In this paper we study the invariant metrizable and projective metrizable problems for the special case of the geodesic flow associated to the canonical connection of a Lie group. We prove that the canonical connection is projectively Finsler metrizable if and only if it is Riemann metrizable. This result means that the structure is rigid in the sense that considering left-invariant metrics, the potentially much larger class of projective Finsler metrizable sprays, corresponding to Lie groups, coincides with the class of Riemann metrizable sprays. Generalisation of these results for geodesic orbit spaces are given.

**2000 Mathematics Subject Classification:** 53B05, 53B40, 70H03, 70H30, 70F17.

**Key words and phrases:** Euler-Lagrange equation; geodesics; metrizable and projective metrizable; Lie group, homogeneous space; geodesic orbit structure.

## 1 Introduction

Lie groups represent a well developed theory of continuous symmetries of mathematical structures, and it is an indispensable tool for modern theoretical physics. The algebraic and differential structures allow us to consider a class of natural objects which have been extensively investigated for over 100 years. Among these one of the most interesting, from differential geometric point of view, is the canonical geodesic structure: it consists of the family of curves given by the 1-dimensional subgroups of the Lie group and their left translated images. The second order differential equation (SODE) associated to this geodesic structure is called *the canonical SODE* and the vector field corresponding to the geodesic flow is the *canonical spray* of the Lie group.

In this paper we investigate the Riemann and Finsler metrizable and projective metrizable of the canonical spray of Lie groups. A spray is called Riemann (resp. Finsler) metrizable, if there exists a Riemann (resp. Finsler) metric such that the geodesics are the solution of the spray. For the more general projective metrizable problem, one seeks for a Riemann (resp. Finsler) metric whose geodesics coincide with the solution of the spray, up to an orientation preserving reparameterization. Recently, several papers have appeared on the metrizable and projective metrizable problems [2, 4, 6, 9]. These articles show that the metrizable and projective metrizable problems are very complex, and even in low dimensional cases, the complete classification is very difficult. Concerning Lie groups, G. Thompson and his co-workers investigated the inverse problem of Lagrangian dynamics for the canonical flow in a series of papers [14, 18, 19]. The problem of the existence of a left-invariant variational principle for the canonical spray was considered in [10, 12], and for invariant second-order differential equations, using the Helmholtz conditions, in [5].

We remark that the notion of a Finsler metric is a generalisation of the Riemannian one. Therefore, every Riemann metrizable spray is also Finsler metrizable. The converse is not true in general. However, in the category of quadratic sprays the situation is different: using Szabó's theorem [16] – stating that for a Finsler metric, admitting a quadratic geodesic spray there exists a Riemannian metric such that the geodesics of the Finsler and the Riemann metrics coincide – the category of Riemann and Finsler metrizable sprays are the same. The projective Finsler metrizability, however, is *essentially different* even in the case of quadratic sprays. This follows as a quadratic spray can be projectively equivalent to a non-quadratic one, and obviously, the latter cannot be Riemann metrizable, because the geodesic spray of Riemann metrics are quadratic. Therefore, the category of projective Finsler metrizable sprays is generally strictly larger than the category of Riemann metrizable sprays, even for quadratic sprays.

The goal of this paper is to investigate the relationship between invariant metrizability, and the invariant projective metrizability of the canonical sprays for the class of Lie groups. In the case of the invariant metrizability problem we ask if there exists a *left-invariant* Riemann (resp. Finsler) metric, such that the geodesics are the solutions of the canonical spray, and in the case of the invariant projective metrizability problem we ask if there exists a left-invariant Riemann (resp. Finsler) metric, such that the geodesics are projectively equivalent to the solutions of the canonical spray. We prove that the canonical connection of a Lie group is invariant projective Finsler metrizable if and only if it is invariant Riemann metrizable. This result shows that the structure is rigid in the sense that considering left-invariant metrics, the potentially much larger class of projective Finsler metrizable sprays, corresponding to Lie groups, coincides with the class of Riemann metrizable sprays.

We consider also homogeneous spaces  $G/H$  with a special geodesic structure. A left invariant geodesic structure on  $G/H$  is called *geodesic orbit structure (g.o. structure)*, if the geodesics can be derived as orbits of 1-parameter subgroups of  $G$ . In V.I. Arnold's terminology these curves are called "relative equilibriums" [1]. We prove that a g.o. structure is projectively invariant Riemann (resp. Finsler) metrizable if and only if it is invariant Riemann (resp. Finsler) metrizable. In the quadratic case we also obtain the rigidity property: the class of projective Finsler metrizable and Riemann metrizable g.o. sprays coincide.

## 2 Differential geometric background of the inverse problem of the calculus of variations

In this section we present the basic objects and tools required for our investigation. More details can be found in [4, 8].

### SODEs, sprays and associated connection

Let  $M$  be a manifold,  $TM$  its tangent bundle and  $\pi$  the natural projection. The map  $J : TTM \rightarrow TTM$  denotes the canonical vertical endomorphism and  $C \in \mathfrak{X}(TM)$  the Liouville vector field. If  $x = (x^i)$  is a local coordinate system on  $M$  and  $(x, y) = (x^i, y^i)$  is the induced coordinate system on  $TM$ , we then have that  $J = dx^i \otimes \frac{\partial}{\partial y^i}$ , and  $C = y^i \frac{\partial}{\partial y^i}$ . Using the vector field  $C$  and Euler's theorem on homogeneous functions, a Lagrangian  $L : TM \rightarrow \mathbb{R}$  is an  $\ell$ -homogeneous function in the  $y$  variable if and only if

$$CF = \ell \cdot F. \tag{2.1}$$

A *spray* is a vector field  $S$  on  $TM \setminus \{0\}$  satisfying the relations  $JS = C$  and  $[C, S] = S$ . The coordinate representation of a spray  $S$  takes the form

$$S = y^i \frac{\partial}{\partial x^i} + f^i(x, y) \frac{\partial}{\partial y^i}, \quad (2.2)$$

where the functions  $f^i(x, y)$  are homogeneous of degree 2 in  $y$ .  $S$  is called quadratic, if  $f^i(x, y)$  are quadratic in  $y$ . The *geodesics* of a spray  $S$  are the curves  $\gamma : I \rightarrow M$  such that  $\dot{\gamma}$  is an integral curve of  $S$ , that is  $S \circ \dot{\gamma} = \ddot{\gamma}$ . The curve  $\gamma$  is a geodesic of (2.2) if and only if it is a solution of the SODE:  $\ddot{x}^i = f^i(x, \dot{x})$ ,  $i = 1, \dots, n$ . Two family of curves are projectively equivalent, if they coincide up to an orientation preserving reparameterization. In this spirit we also call two SODEs (or sprays) projectively equivalent if their solutions (as parametrised curves) are projectively equivalent. It is easy to show that two sprays  $S$  and  $\bar{S}$  are projectively equivalent if and only if there exists a 1-homogeneous function  $\mathcal{P} = \mathcal{P}(x, y)$  such that  $\bar{S} = S - 2\mathcal{P}C$ .

A (*non-linear*) *connection* on  $M$  is a type (1-1) tensor field  $\Gamma$  on  $TM$  such that  $J\Gamma = J$  and  $\Gamma J = -J$ . If  $\Gamma$  is a connection, then  $\Gamma^2 = id_{TTM}$  and the eigenspace corresponding to the eigenvalue  $-1$  is the vertical space. The eigenspace  $H$  corresponding to the eigenvalue  $+1$  is called the *horizontal space*. In the sequel we will write  $h = \frac{1}{2}(I + \Gamma)$  and  $v = \frac{1}{2}(I - \Gamma)$ , for the *horizontal* and *vertical* projectors. If  $S$  is a spray, then  $\Gamma = [J, S]$  is a connection which is called the *connection associated to  $S$* .

### The Euler-Lagrange ODE and PDE

It is well known that if a *Lagrangian*  $L : TM \rightarrow \mathbb{R}$  is regular, that is  $\det\left(\frac{\partial^2 L}{\partial y^i \partial y^j}\right) \neq 0$ , then the 2-form  $\Omega_L := dd_J L$  has maximal rank. For a regular Lagrangian  $L$ , homogeneous of order 2, the vector field  $S$  on  $TM$  defined by the equation

$$i_S \Omega_L + d(d_C L - L) = 0 \quad (2.3)$$

(where  $d_C L$  denotes the Lie derivative of  $L$  with respect to  $C$ ) is a spray, and the paths of  $S$  are the solutions of the Euler-Lagrange ODE.

Let us fix a spray  $S$  on the manifold  $M$ . Then, to every Lagrangian  $L$ , a scalar 1-form  $\omega_L = i_S \Omega_L + d(d_C L - L)$ , called the *Euler-Lagrange form* can be associated. Then the Euler-Lagrange equation can be written as

$$\omega_L = 0. \quad (2.4)$$

The equation (2.4) holds if and only if the paths of the spray  $S$  are the solutions of the Euler-Lagrange equation associated to  $L$ . We remark, that when  $S$  is given, (2.4) is a system of second order PDE on  $L$  called the *Euler-Lagrange PDE*.

## 3 Metrizable and projective metrizable

The metrizable and projective metrizable problems can be formulated as follows:

**Definition 3.1.** *A spray  $S$  is*

- 1) *Riemann (resp. Finsler) metrizable, if there exists a Riemann (resp. Finsler) metric whose geodesics coincide with the geodesics of  $S$ .*

- 2) projective Riemann (resp. Finsler) metrizable, if there exists a Riemann (resp. Finsler) metric whose geodesics are projectively equivalent with the geodesics of  $S$ .

Both the metrizability and projective metrizability problems can be formulated in terms of a system of partial differential equations which is composed of the appropriate homogeneity condition and the Euler-Lagrange PDE equations on the energy function. Indeed, the following holds.

**Proposition 3.2.** *A spray  $S$  on a manifold  $M$  is*

- 1) Riemann (resp. Finsler) metrizable if and only if there exists a quadratic (resp. 2-homogeneous) function  $E: TM \rightarrow \mathbb{R}$ , such that the matrix field  $(\frac{\partial^2 E}{\partial y^i \partial y^j})$  is positive definite on  $TM \setminus \{0\}$  and (2.4) is satisfied with  $L := E$ .
- 2) projectively Riemann (resp. Finsler) metrizable if and only if there exists a quadratic (resp. 2-homogeneous) function  $\widehat{E}: TM \rightarrow \mathbb{R}$ , such that the matrix field  $(\frac{\partial^2 \widehat{E}}{\partial y^i \partial y^j})$  is positive definite on  $TM \setminus \{0\}$  and (2.4) is satisfied with  $L := \sqrt{2\widehat{E}}$ .

*Proof.* For 1) we remark that the Riemann (resp. Finsler) metric  $g = g_{ij}dx^i \otimes dx^j$  exists if and only if the associated energy function  $E = g_{ij}y^i y^j / 2$  exists. Thus a function  $E$  is the Riemann (resp. Finsler) energy function corresponding to  $S$  if and only if it satisfies the conditions formulated in point 1) of the proposition.

For 2) we note that the spray  $S$  is projectively Riemann (resp. Finsler) metrizable if and only if there exists a Riemann (resp. Finsler) metrizable spray  $\widehat{S}$  which is projectively equivalent to  $S$ . In that case there exists a 1-homogeneous function  $\mathcal{P}$ , such that  $\widehat{S} = S - 2\mathcal{P}C$ . Let us denote by  $\widehat{E} = \widehat{g}_{ij}y^i y^j / 2$  the energy function of  $\widehat{S}$  and let  $\widehat{F} = \sqrt{2\widehat{E}}$  be the associated Finsler function. It is well known that  $\widehat{F}$  is invariant with respect to the parallel translation associated to the connection  $\widehat{\Gamma}$  and therefore we have  $d_{\widehat{h}}\widehat{F} = 0$  where  $\widehat{h} = h - \mathcal{P}J - d_J\mathcal{P} \otimes C$  (see [3], section 4). Using the 1-homogeneity of  $\widehat{F}$ , we get

$$0 = d_{\widehat{h}}\widehat{F} = d_h\widehat{F} - d_J(\mathcal{P}\widehat{F}). \quad (3.1)$$

Substituting  $S$  into the above formula and using the 1-homogeneity of  $\mathcal{P}$  and  $\widehat{F}$  we get  $d_{\widehat{h}}\widehat{F}(S) = S\widehat{F} - 2\mathcal{P}\widehat{F} = 0$  and therefore the projective factor is

$$\mathcal{P} = \frac{1}{2\widehat{F}}S\widehat{F}. \quad (3.2)$$

Replacing  $\mathcal{P}$  in (3.1) and using the Frölicher-Nijenhuis formalism we get

$$0 = 2d_h\widehat{F} - d_J(\frac{1}{\widehat{F}}\mathcal{L}_S\widehat{F}\widehat{F}) = d_{\Gamma+J}\widehat{F} - d_Jd_S\widehat{F} = d_{[J,S]}\widehat{F} + d\widehat{F} - d_Jd_S\widehat{F} = -i_Sdd_J\widehat{F}. \quad (3.3)$$

On the other hand, since  $\widehat{F}$  is a 1-homogeneous function, we have  $C\widehat{F} = \widehat{F}$  and  $dd_C\widehat{F} - d\widehat{F} = 0$ . Therefore the Euler-Lagrange form is  $\omega_{\widehat{F}} = i_Sdd_J\widehat{F} + dd_C\widehat{F} - d\widehat{F} = i_Sdd_J\widehat{F}$ . Comparing this with (3.3) we get that  $\omega_{\widehat{F}} = 0$ . This shows that the conditions of 2) are necessary conditions.

Conversely, let us suppose that the function  $\widehat{E}$  satisfies the conditions of 2). If we define  $\mathcal{P}$  by (3.2), then  $\widehat{S} = S - 2\mathcal{P}C$  is projectively equivalent to  $S$ , and it is not difficult to show that the spray associated to  $\widehat{E}$  is given by  $\widehat{S}$ .  $\square$

We note, that a coordinate version of 2) was first proved by A. Rapcsák in [15] and a coordinate free version of this the statement was given by J. Szilasi and Sz. Vattamány in [17]. Here we have presented a different approach.

**Remark 3.3.** For a given spray  $S$ , the second order partial differential equation  $i_S dd_J F = 0$  is called the *Rapcsák equation*. As we have seen in equation (3.3), for 1-homogeneous functions the Rapcsák equation is equivalent to the Euler-Lagrange equation.

**Proposition 3.4.** *Let  $S$  be a spray and  $L$  be a Lagrangian. If  $L$  is invariant with respect to  $S$ , then we have*

$$\omega_L = 0 \quad \Leftrightarrow \quad d_h L = 0 \quad (3.4)$$

*Proof.* If  $L$  is invariant with respect to  $S$ , that is  $S(L) = d_S L = 0$  then, using the Frölicher-Nijenhuis calculus, we get

$$\begin{aligned} \omega_L &= i_S dd_J L + dd_C L - dL = d_S d_J L - di_S d_J L + dd_C L - dL \\ &= d_{[S, J]} L + d_J d_S L - d(d_{JS} - d_J i_S + i_{[J, S]}) L + dd_C L - dL \\ &= -d_{\Gamma} L - dL = -d_{\Gamma+I} L = -2d_h L \end{aligned}$$

□

**Corollary 3.5.** *Let  $S$  be a spray, let  $L$  be a non-zero invariant function with respect to  $S$  and  $k \neq 0$ . Then  $L$  is a solution of the Euler-Lagrange equation associated to  $S$  if and only if  $L^k$  is a solution of the Euler-Lagrange equation associated to  $S$ .*

*Proof.* Using Theorem 3.4 we know that equation  $\omega_L = 0$  is equivalent to  $d_h L = 0$  and equation  $\omega_{L^k} = 0$  is equivalent to  $d_h L^k = 0$ . Moreover, since

$$d_h(L^k) = kL d_h L, \quad (3.5)$$

we have  $d_h(L^k) = 0$  if and only if  $d_h L = 0$  holds. □

## 4 Invariant metrizable and projective metrizable of the canonical flow of Lie groups

Let  $G$  be a finite dimensional Lie group. We denote by  $\lambda_g: G \rightarrow G$  the left translation of  $G$  defined by  $\lambda_g(\hat{g}) = g\hat{g}$ . Let  $x = (x^1, \dots, x^n)$  be a coordinate system on  $G$  and  $(x, y)$  be the usual associated standard coordinate system on  $TG$  where  $y = (y^1, \dots, y^n)$  with  $y^i = dx^i$ . We will be interested in investigating left invariant structures on  $TG$ . It is thus more convenient to introduce a kind of "semi-invariant" coordinate system using the left trivialisation  $TG \cong G \times \mathfrak{g}$ . Indeed, for every  $g \in G$  the tangent space  $T_g G$  is isomorphic to  $\mathfrak{g}$  by the tangent map of the left translation  $(\lambda_g^{-1})_* : T_g G \rightarrow \mathfrak{g} = T_e G$ . Therefore, one can introduce a left invariant  $\mathfrak{g}$ -valued differential form  $\alpha: TG \rightarrow \mathfrak{g}$ , known as the Maurer-Cartan form, defined by  $\alpha = (\lambda_g^{-1})_* dg$ . The corresponding *semi-invariant coordinate system* is given by  $(x^i, \alpha^i)$  where  $\alpha^i = (\lambda_{x,*}^{-1})^i_j dx^j$ . The left invariant coordinate system  $(x, \alpha)$  induces coordinates on the second tangent bundle  $TTG$  which will be denoted by

$$(x^i, \alpha^i, X^i, A^i) \simeq X^i \frac{\partial}{\partial x^i} \Big|_{(x, \alpha)} + A^i \frac{\partial}{\partial \alpha^i} \Big|_{(x, \alpha)},$$

We note that, by using a simplified notation  $(x^i, \alpha^i, X^i, A^i) \simeq (x, \alpha, X, A)$ , the coordinates  $x = (x^i)$ , and therefore the coordinates  $X = (X^i)$  are not left invariant, but  $\alpha = (\alpha^i)$  and therefore the corresponding  $A = (A^i)$ , are: the left translation by a group element  $g$  induces on  $TTG$  the following action:  $(\lambda_g)_{**}(x, \alpha, X, A) = (gx, \alpha, gX, A)$ .

**Definition 4.1.** A vector field  $X \in \mathfrak{X}(G)$  is called left invariant, if  $(\lambda_g)_*X = X$  for every  $g \in G$ . Similarly, a function  $L: TG \rightarrow \mathbb{R}$  is called left invariant, if  $L \circ \lambda_{g*} = L, \forall g \in G$ .

Using the semi-invariant coordinate system  $(x^i, \alpha^i)$ , the function  $L: TG \rightarrow \mathbb{R}$  is left invariant if and only if its value does not depend on the  $x$ -coordinates, that is

$$\frac{\partial L}{\partial x^i} \equiv 0, \quad i = 1, \dots, n. \quad (4.1)$$

**Definition 4.2.** The canonical geodesic structure on a Lie group  $G$  is given by the 1-parameter subgroups and their left (or right) translated images. The canonical SODE of  $G$  is the SODE corresponding to the canonical flow.

The main geometric objects associated to the canonical flow (spray, horizontal and vertical projections etc.) were calculated in [12] and [14]. Here we just present the essential results needed for our purpose. More about the computation of these objects can be found in the above mentioned papers.

Using matrix representation, in a standard coordinate system  $(x, y)$ , the canonical SODE can be described by

$$\ddot{x} = \dot{x}x^{-1}\dot{x}. \quad (4.2)$$

The canonical spray corresponding to (4.2) in the semi-invariant coordinate system  $(x, \alpha)$  is given by  $S \in \mathfrak{X}(TG)$ , where

$$S_{(x,\alpha)} = x\alpha \frac{\partial}{\partial x} \Big|_{(x,\alpha)}. \quad (4.3)$$

The vertical and horizontal projectors are defined as follows. For every  $(x, \alpha) \in TG$  we have

$$v \left( xa \frac{\partial}{\partial x} \Big|_{(x,\alpha)} + b \frac{\partial}{\partial \alpha} \Big|_{(x,\alpha)} \right) = \left( \frac{1}{2}[a, \alpha] + b \right) \frac{\partial}{\partial \alpha} \Big|_{(x,\alpha)}, \quad (4.4)$$

$$h \left( xa \frac{\partial}{\partial x} \Big|_{(x,\alpha)} + b \frac{\partial}{\partial \alpha} \Big|_{(x,\alpha)} \right) = xa \frac{\partial}{\partial x} \Big|_{(x,\alpha)} - \frac{1}{2}[a, \alpha] \frac{\partial}{\partial \alpha} \Big|_{(x,\alpha)}. \quad (4.5)$$

We have the following

**Proposition 4.3.** (Canonical invariant Euler-Lagrange system)

A Lagrangian  $L: TG \rightarrow \mathbb{R}$  is a left-invariant solution to the Euler-Lagrange equation associated to the canonical spray the Lie group  $G$ , if and only if the system

$$\frac{\partial L}{\partial x^i} = 0, \quad i = 1, \dots, n \quad (4.6)$$

$$[a, \alpha]^i \frac{\partial L}{\partial \alpha^i} = 0, \quad \forall a \in \mathfrak{g}, \quad (4.7)$$

is satisfied.

*Proof.* The equation (4.6) expresses the left invariant property of  $L$ . Moreover, from the local expression (4.3) of the canonical spray  $S$  it is clear, that if the Lagrangian  $L$  is left invariant, then

$$d_S L = 0. \quad (4.8)$$

Using Proposition 3.4 we get, that the Euler-Lagrange PDE is satisfied if and only if  $d_h L = 0$ . Using (4.5) we have

$$d_h L = 0 \quad \Leftrightarrow \quad xa \frac{\partial L}{\partial x} - \frac{1}{2}[a, \alpha] \frac{\partial L}{\partial \alpha} = 0, \quad a \in \mathfrak{g}.$$

and from the left invariant property we get the later is identically zero if and only if (4.7) is satisfied.  $\square$

The following statement holds.

**Proposition 4.4.** *The canonical spray  $S$  of a Lie group is left invariant projectively Riemann (resp. Finsler) metrizable if and only if it is left invariant Riemann (resp. Finsler) metrizable.*

*Proof.* It is clear that if  $S$  is Riemann (resp. Finsler) metrizable, then it is also projectively Riemann (resp. Finsler) metrizable. Conversely, let us suppose that  $S$  is projectively Riemann (resp. Finsler) metrizable. Then, according to Proposition 3.2, there exists a left invariant (quadratic, resp. 2-homogeneous) function  $\widehat{E} : TG \rightarrow \mathbb{R}$  such that the matrix field  $(\frac{\partial^2 \widehat{E}}{\partial \alpha_i \partial \alpha_j})$  is positive definite on  $TM \setminus \{0\}$  and  $\widehat{F} := \sqrt{2\widehat{E}}$  satisfies the Euler-Lagrange PDE associated to  $S$ . Because of the left invariance we have  $d_S L = 0$  and, using Corollary 3.5, we get that  $\widehat{E} := \frac{1}{2}(\widehat{F})^2$  is also a solution of the Euler-Lagrange PDE associated to  $S$ . Then, according to Proposition 3.2, the given  $\widehat{E}$  is the energy function of a Riemann (resp. Finsler) metric which implies that  $S$  is Riemann (resp. Finsler) metrizable.  $\square$

We have the following

**Theorem 4.5.** *The canonical spray of a Lie group is left invariant projectively Finsler metrizable if and only if it is left invariant Riemann metrizable.*

*Proof.* In one direction the statement is trivial. If the canonical spray is Riemann metrizable, then it is trivially Finsler metrizable and also projectively Finsler metrizable.

Let us consider the converse statement, and suppose that the canonical spray  $S$  is projectively Finsler metrizable. Then, according to Proposition 4.4, it is also Finsler metrizable. Since  $S$  is quadratic, it follows that the associated connection is linear. Hence, the Finsler metrizability induces the existence of a Berwald metric on the Lie group. Using Szabó's theorem which states that for every Berwald metric there exists a Riemannian metric such that the geodesics of the Berwald and Riemannian metrics are the same (cf. [16]), we get that the canonical spray is Riemann metrizable.  $\square$

Using Proposition 4.3 we obtain the following.

**Corollary 4.6.** *The canonical spray of a Lie group  $G$  is left invariant (Riemann, Finsler, projectively Riemann or projectively Finsler) metrizable if and only if there exists a scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$  such that*

$$\langle [a, \alpha], \alpha \rangle = 0 \tag{4.9}$$

for every  $a, \alpha \in \mathfrak{g}$ .

*Proof.* An invariant Riemannian metric induces a scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$ . Using the coordinate system  $(x, \alpha)$  on  $TG \simeq G \times \mathfrak{g}$ , the associated energy function is given by  $E : G \times \mathfrak{g} \rightarrow \mathbb{R}$ , where  $E(x, \alpha) = \langle \alpha, \alpha \rangle$ . The Euler-Lagrange equation (4.7) then implies (4.9).  $\square$

**Remark 4.7.** We want to draw attention to few interesting phenomena. First of all, although the canonical spray of a Lie group is a very natural object, it is not true that it is always metrizable. In [19] there are several examples to Lie groups and Lie algebras where the canonical spray is non metrizable.

Secondly, despite the fact that the canonical spray is left (and also right) invariant, and the Euler-Lagrange equation inherits the symmetries of the Lagrangian, it is not true that the ‘metrizability’ property means automatically ‘metrizability by a left invariant metric’.

Indeed, for example the 3-dimensional Heisenberg group  $\mathbb{H}_3$  is not metrizable or projectively metrizable with an invariant Riemann (or Finsler) metric [12]. However, since the curvature tensor vanishes identically, the canonical spray is metrizable. The corresponding (non invariant) Riemannian metric is given by  $g = dx^2 + dy^2 + (dz - dx - dy)^2$ , (see [19]).

Theorem 4.5 shows that the geometric structure associated to the canonical spray of a Lie group has a certain rigidity property. The potentially much larger class of Lie groups where the canonical spray is projectively equivalent to a Finsler geodesic structure, actually coincides with the class of invariant Riemann metrizable sprays. We note that this property relies heavily on the fact that, 1) the canonical spray is quadratic, and 2) the Lie derivative of a left invariant Lagrange function on  $G$ , with respect to the canonical spray, is identically zero. The second property is not true in general for an arbitrary left invariant spray. However, interesting generalisation can be obtained by considering the class of homogeneous spaces. We consider this in the next section.

## 5 Invariant metrizability and projective metrizability of a geodesic orbit structures of homogeneous spaces

Let  $M$  be a connected differentiable manifold on which the Lie transformation group  $G$  acts transitively. Let us fix an origin  $o \in M$  and denote by  $H$  the stabiliser of  $o \in M$  in the group  $G$  and by  $\pi : G \rightarrow G/H$  the projection map. As usual we call  $H$  the isotropy group of the homogeneous space  $G/H$ . Then  $M$  is isomorphic to the factor space  $G/H$  with origin  $H$  and its tangent space at  $o \in M$  is isomorphic to  $\mathfrak{g}/\mathfrak{h}$ , where  $\mathfrak{g}$  and  $\mathfrak{h}$  are the Lie-algebras of the Lie groups  $G$  and  $H$  respectively. The action of  $G$  on  $M$  is determined by the map

$$\lambda : (g, m) \mapsto \lambda_g m = g \cdot m : G \times M \rightarrow M.$$

Geodesic structures, sprays, metrics, Lagrangians on  $M$  are called *invariant*, if they are invariant with respect to the action of  $G$ . It is clear, that invariant sprays, metrics, and Lagrangians can be characterised by their values on  $T_o M$ .

In the most interesting cases, the algebraic structures and invariant geometric structures are intimately related: the geodesics are the image of the 1-parameter subgroups of the group  $G$  and their left translated images. More precisely we have the following

**Definition 5.1.** *A geodesic  $\gamma(t)$  emanated from the origin  $o \in M$  is called homogeneous, if there exists  $X_\gamma \in \mathfrak{g}$ , such that  $\gamma(t)$  is the orbit of the 1-parameter subgroup  $\{\exp tX_\gamma, t \in \mathbb{R}\}$  of  $G$ , that is*

$$\gamma(t) = \lambda_{\exp tX_\gamma} o = (\exp tX_\gamma) \cdot o. \quad (5.1)$$

*The Lie algebra element  $X_\gamma \in \mathfrak{g}$  is called geodesic vector associated to the direction  $\dot{\gamma}(0) \in T_o M$ .*

**Definition 5.2.** *A left invariant geodesic structure is called geodesic orbit structure (g.o. structure), if any geodesic  $\gamma(t)$  emanated from the origin  $o \in M$  is homogeneous (in the sense of Definition 5.1). A spray is called geodesic orbit spray (g.o. spray), if it corresponds to a g.o. structure.*

Homogeneous geodesics are called in V.I. Arnold's terminology "relative equilibriums" [1].

**Definition 5.3.** *A map  $\sigma : T_o M \rightarrow \mathfrak{g}$  is called homogeneous lift<sup>1</sup> if the following conditions are satisfied:*

---

<sup>1</sup>In [13] the terminology *horizontal lift* was used however, in Finsler geometry, this terminology is wildly used for a different object.

1.  $\pi_* \circ \sigma = id_{T_oM}$ .
2.  $\sigma$  is 1-homogeneous, that is  $\sigma(\kappa \cdot v) = \kappa \sigma(v)$ , for every  $v \in T_oM$  and  $\kappa \in \mathbb{R}$ .
3.  $\sigma$  is  $\text{Ad}(H)$ -invariant, that is  $\sigma(\lambda_h v) = \text{Ad}_h \sigma(v)$  for all  $h \in H$  and  $v \in T_oM$ .

The homogeneous lift  $\sigma$  is called  $C^\infty$ -differentiable if it is continuous on  $T_oM$  and  $C^\infty$ -differentiable on  $T_oM \setminus \{0\}$ .

It is clear that any g.o. spray determines a  $C^\infty$ -differentiable homogeneous lift by associating to  $v \in T_oM$  its geodesic vector  $X = \sigma(v)$  and vice versa, every homogeneous lift determines a g.o. spray by left translations.

**Lemma 5.4.** *Invariant functions are constant along the geodesics of a g.o. spray.*

*Proof.* Because of the invariance, it is enough to show this property for geodesics emanated from the origin  $o \in M$ . Let  $S$  be a geodesic orbit spray on a homogeneous space  $M = G/H$ . For a geodesic  $\gamma(t)$  emanated from the origin  $o \in M$  there exists a geodesic vector  $X_\gamma \in \mathfrak{g}$  such that we have (5.2). Using the 1-parameter subgroup property we can write

$$\gamma(t_0 + t) = \lambda_{\exp(t_0+t)X_\gamma} o = \lambda_{\exp t_0 X_\gamma} \lambda_{\exp t X_\gamma} o \quad (5.2)$$

and therefore we have

$$\dot{\gamma}(t_0) = \left. \frac{d}{dt} \right|_{t=0} \gamma(t_0 + t) = \left. \frac{d}{dt} \right|_{t=0} \lambda_{\exp t_0 X_\gamma} \lambda_{\exp t X_\gamma} o = (\lambda_{\exp t_0 X_\gamma})_* \dot{\gamma}(0). \quad (5.3)$$

Let  $L: TM \rightarrow \mathbb{R}$  be an invariant function. For every  $p \in M$ ,  $v \in T_pM$  and  $g \in G$  we have  $L(v) = L(\lambda_{g,*}v)$ . Applying this with  $v = \dot{\gamma}(0)$  and  $g = \exp t_0 X_\gamma$  we get from (5.3) that  $L(\dot{\gamma}(t_0)) = L(\dot{\gamma}(0))$  that is  $L$  is constant on the geodesics.  $\square$

Using Lemma 5.4 we can obtain the generalisation of Proposition 4.4:

**Proposition 5.5.** *A g.o. spray is projectively Riemann (resp. Finsler) invariant metrizable if and only if it is invariant Riemann (resp. Finsler) metrizable.*

*Proof.* Let  $S$  be a g.o. spray. As Lemma 5.4 shows, if  $L: TM \rightarrow \mathbb{R}$  is a  $G$  invariant Lagrangian, then  $L$  is constant along the geodesics, that is along the integral curves of  $S$ . Consequently we have  $d_S L = 0$ . Using Corollary 3.5 and the same argument that was used for Proposition 4.4 we can obtain the proof of the proposition.  $\square$

We remark that the connection determined by a g.o. spray is not necessarily linear, therefore it is not true in general that the Finsler metrizable entail the Riemann metrizable as it was the case for the canonical spray of Lie groups. However, if the g.o. spray is quadratic then the associated connection is linear. Therefore we can use Szabó's theorem and similarly to Theorem 4.5 we can get the following

**Theorem 5.6.** *A quadratic g.o. spray is left invariant projectively Finsler metrizable if and only if it is invariant Riemann metrizable.*

*Remark.* A different invariant metrizable concept of the  $G/H$  structure is considered in [7] by S. Deng and Z. Hou where the  $G/H$  structure is called invariant metrizable if there exists an invariant metric on it. The invariant metrizable (and projective metrizable) of a g.o. structure or g.o. spray is however more subtle, because in this case not only the  $G/H$  homogeneous space, but also the geodesic structure is fixed and we want to metrize both. It may happen that the g.o. structure on a homogeneous space  $G/H$  is not invariant metrizable but  $G/H$  is. To illustrate this phenomenon, let us consider the following

**Example.** Let us consider  $M = \mathbb{R}^2$  as the homogeneous space  $\mathbb{R}^2 = G/H$ , where  $G = ASO(2)$  is the euclidean transformation group, and  $H = SO(2)$  is the rotation subgroup.  $H$  is the stabiliser of the point  $o = (0, 0) \in \mathbb{R}^2$ . The matrix representation can be given by

$$G = \left\{ \begin{pmatrix} \cos t & -\sin t & x_1 \\ \sin t & \cos t & x_2 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad H = \left\{ \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

and the corresponding Lie algebras are

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & -a_3 & a_1 \\ a_3 & 0 & a_2 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad \mathfrak{h} = \left\{ \begin{pmatrix} 0 & -a_3 & 0 \\ a_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

We have  $G/H \simeq \mathbb{R}^2$ , and the projection map is

$$\pi : G \rightarrow G/H \simeq \mathbb{R}^2, \quad \begin{pmatrix} \cos t & -\sin t & x_1 \\ \sin t & \cos t & x_2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\pi} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} \sim (x_1, x_2). \quad (5.4)$$

Using the identification " $\sim$ " introduced in (5.4), the action of  $G$  can be interpreted as a matrix multiplication:  $\lambda_g \hat{x} = g \cdot \hat{x}$ . Since the homogeneous lift for this  $G/H$  structure must be a rotation-invariant 1-homogeneous map, it is of the form

$$\sigma_\kappa : T_o \mathbb{R}^2 \rightarrow \mathfrak{g}, \quad (v_1, v_2) \xrightarrow{\sigma_\kappa} \begin{pmatrix} 0 & -\kappa \sqrt{v_1^2 + v_2^2} & v_1 \\ \kappa \sqrt{v_1^2 + v_2^2} & 0 & v_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad (5.5)$$

where  $\kappa \in \mathbb{R}$  is an arbitrary (but fixed) constant. The geodesic equation corresponding to the homogeneous lift (5.5) can be given as follows: a curve  $t \rightarrow \gamma(t)$  is a geodesic if and only if

$$\ddot{\gamma} = \sigma_\kappa(\dot{\gamma})\dot{\gamma}. \quad (5.6)$$

Indeed, let  $\gamma(t)$  be the geodesic in the direction  $v \in T_o M$  and let  $X = \sigma_\kappa(v)$  be the geodesic vector corresponding to  $v$ . Using matrix representation we get  $\gamma(t) = e^{t\sigma_\kappa(v)} o$  and therefore for its second derivative we can find  $\ddot{\gamma}(t) = \sigma_\kappa(v)\sigma_\kappa(v)e^{t\sigma_\kappa(v)} o = \sigma_\kappa(v)\dot{\gamma}(t)$ . Since  $v = \dot{\gamma}(0)$  and the tangent vectors are translations invariant, we can obtain (5.6). The system of differential equations of geodesics is

$$\begin{cases} \ddot{x}_1 = -\kappa \sqrt{\dot{x}_1^2 + \dot{x}_2^2} \dot{x}_2, \\ \ddot{x}_2 = \kappa \sqrt{\dot{x}_1^2 + \dot{x}_2^2} \dot{x}_1. \end{cases} \quad (5.7)$$

If  $\kappa = 0$ , then the curvature tensor is identically zero and the geodesics are straight lines. The corresponding g.o. structure is metrizable: the usual euclidean metric ( $g_{ij} = \delta_{ij}$ ) metricises it. If  $\kappa \neq 0$ , then the geodesics are non-reversible (that is  $\gamma_{p,-v}(t) \neq \gamma_{p,v}(-t)$ ) and the curvature tensor is non-zero:

$$R_{(x,y)}^\kappa \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) = -\kappa^2 y_2 \frac{\partial}{\partial y_1} + \kappa^2 y_1 \frac{\partial}{\partial y_2}.$$

It is easy to show, using the metrizable criteria of [4] or [11], that the corresponding g.o. structure is not metrizable. Therefore it is not invariant metrizable as well.

This examples illustrates clearly the difference between the concept of the metrizable of the  $G/H$  structure and the metrizable of the g.o. structure: the homogeneous space  $\mathbb{R}^2 = ASO(2)/SO(2)$  is invariant metrizable (for example an invariant metric is given by the standard euclidean metric) however the g.o. structure determined by the homogeneous lift (5.5) is not, unless  $\kappa = 0$ .

## Acknowledgments

This work has been supported by the EU FET FP7 BIOMICS project, contract number CNECT-318202.

## References

- [1] Arnold, V.I.: *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaites*, Ann. Inst. Fourier, 16 (1960), 319–361.
- [2] Bryant, R.L., Dunajski, M., Eastwood, M.: *Metrisability of two-dimensional projective structures*, J. Differential Geom. 83, Number 3 (2009), 465–500.
- [3] Bucataru, I., Muzsnay, Z.: *Projective and Finsler metrizable sprays: parameterization-rigidity of the geodesics*, Int. J. Math., 23, no. 9, (2012) 1250099.
- [4] Bucataru, I., Muzsnay, Z.: *Metrizable isotropic second-order differential equations and Hilbert's fourth problem*, J. Aust. Math. Soc, 97, 01 (2014), 27–47.
- [5] Crampin, M., Mestdag, T.: *The inverse problem for invariant Lagrangians on a Lie group*, J. Lie Theory 18: (2) (2008), 471–502.
- [6] Crampin, M., Mestdag, T., Saunders, D.J. : *Hilbert forms for a Finsler metrizable projective class of sprays*, Diff. Geom. Appl. 31, 1, (2013), 63–79.
- [7] Deng, S., Hou, Z.: *Invariant Finsler metrics on homogeneous manifolds*, J. Phys. A: Math. Gen. 37, (2004), 8245–8253.
- [8] Grifone, J., Muzsnay, Z.: *Variational Principles For Second-Order Differential Equations*, World Scientific, Singapore, 2000.
- [9] Matveev, V.S.: *Geodesically equivalent metrics in general relativity*, J.Geom.Phys. 62, (2012), 675–691.
- [10] Mestdag, T.: *Relative equilibria of invariant Lagrangian systems on a Lie Group*, Geometric Methods in Mechanics and Field Theory. Gent: Academia Press, (2007), 91–114.
- [11] Muzsnay, Z.: *The Euler-Lagrange PDE and Finsler metrizable sprays*, Houston Journal of Mathematics, 32 no. 1, (2006) pp. 79-98.
- [12] Muzsnay, Z.: *An invariant variational principle for canonical flows on Lie groups*, J. Math. Phys. 46 (2005), no. 11, 112902, pp. 11.
- [13] Muzsnay, Z., Nagy, P.T.: *Invariant Shen connections and geodesic orbit spaces*, Period. Math. Hungar. 51 (2005), no. 1, 37-51.
- [14] Muzsnay, Z., Thompson, G.: *Inverse problem of the calculus of variations on Lie groups*, Diff. Geom. Appl. 23 (2005), no. 3, 257–281.
- [15] Rapcsák, A. *Über die bahntreuen Abbildungen metrischer Räume*, Publ. Math. Debrecen 8 (1961), 285–290.
- [16] Szabó, Z.: *Positive definite Berwald spaces*, Tensor, New Ser., 35 (1981), 25–39.

- [17] Szilasi, J., Vattamány Sz.: *On the Finsler-metrizabilities of spray manifolds*, Period. Math. Hun. Vol. 44 (1), (2002), pp. 81–100.
- [18] Thompson, G.: *Variational connections on Lie groups*, Diff. Geom. Appl., 18 (2003), 255–270.
- [19] Ghanam, R.; Hindeleh, F.; Thompson, G. *Bi-invariant and noninvariant metrics on Lie groups* J. Math Phys; Vol. 48 Issue 10, p.102903.

Ioan Bucataru  
Faculty of Mathematics, Alexandru Ioan Cuza University  
700506, Bulevardul Carol I 11, Iași, Romania  
*E-mail:* [bucataru@uaic.ro](mailto:bucataru@uaic.ro)

Tamás Milkovszki  
Institute of Mathematics, University of Debrecen,  
H-4032 Debrecen, Egyetem tér 1, Hungary  
*E-mail:* [milkovszki@science.unideb.hu](mailto:milkovszki@science.unideb.hu)

Zoltán Muzsnay  
Institute of Mathematics, University of Debrecen,  
H-4032 Debrecen, Egyetem tér 1, Hungary  
*E-mail:* [muzsnay@science.unideb.hu](mailto:muzsnay@science.unideb.hu)