Holonomy distribution and degree of metrizability of a SODE

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Calculus of variations:

- scalar product $\langle v, v\rangle = E(v)$ Riemann: $\langle v, v\rangle_x$ Finsler: $\langle v, v\rangle_{(x,y)}$
- $I[\gamma] = \int_\gamma E(\gamma, \dot\gamma)$
- $\dfrac{d}{dt}\dfrac{\partial E}{\partial \dot x^i} - \dfrac{\partial E}{\partial x^i} = 0, \quad \Rightarrow \quad \dot x^i \dfrac{\partial^2 E}{\partial x^i \partial \dot x^j} + \ddot x^i \dfrac{\partial^2 E}{\partial \dot x^i \partial \dot x^j} - \dfrac{\partial E}{\partial x^j} = 0 \quad \Rightarrow \quad \ddot x^i = f^i(x, \dot x),$

Metric	$\Rightarrow$	SODE

The inverse problem: metrizability

Metric	$\overset{?}{\longleftarrow}$	SODE

Remark: metrizability $\Longleftrightarrow$ Paolo: dynamical potential?

Geometric tools assotiated to a SODE:

· SODE $\Longrightarrow$ Spray: vectorfield on TM $\quad S = \big(x^i, y^i, y^i, f^i(x,y)\big)$

· Path of the spray: γ $\quad S_{\dot\gamma} = \ddot\gamma \quad \Longleftrightarrow \quad \dfrac{d^2x^i}{dt^2} = f^i\left(x, \dfrac{dx}{dt}\right).$

· Parallel translation...

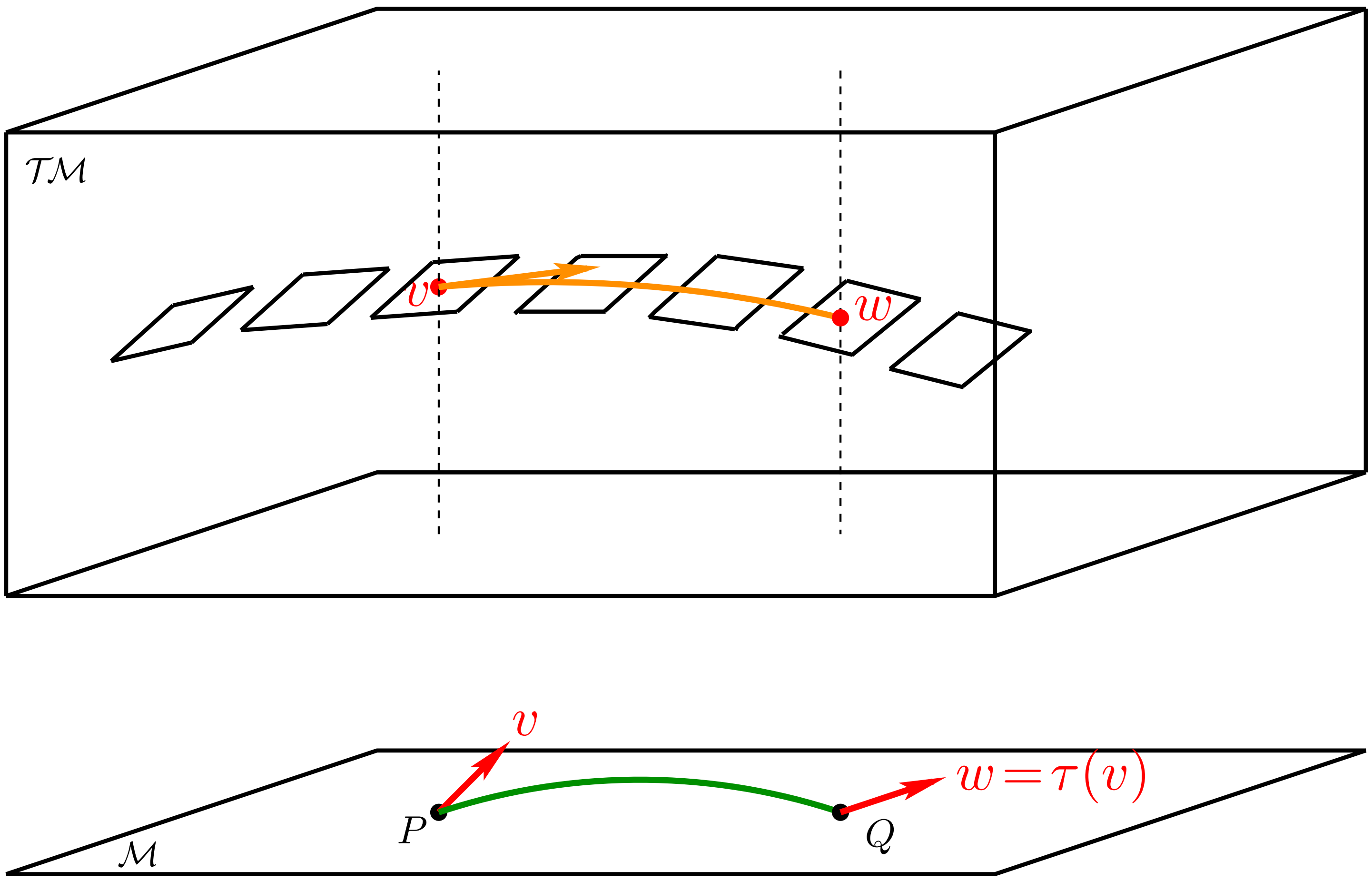
The Euler-Lagrange PDE system

$$\ddot{x}^i = f^i(x,\dot{x}) \quad \Rightarrow \quad \begin{cases} \omega_E = y^i \dfrac{\partial^2 E}{\partial x^i \partial y^j} + f^i \dfrac{\partial^2 E}{\partial y^i \partial y^j} - \dfrac{\partial E}{\partial x^j} = 0, \\ y^i \dfrac{\partial E}{\partial y^i} - 2E = 0, \end{cases}$$

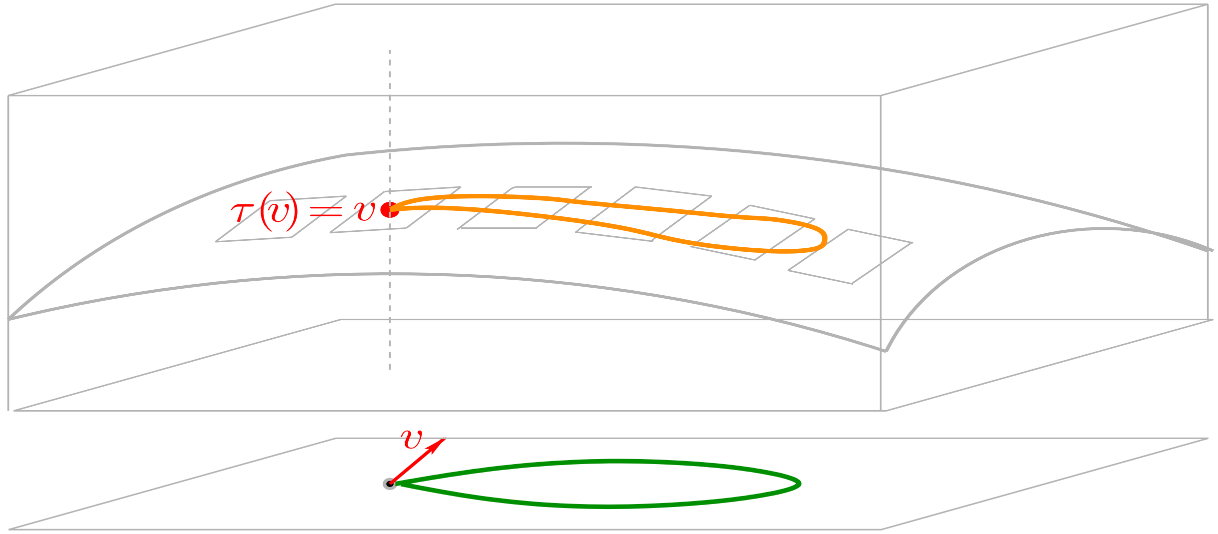
Euler-Lagrange functions

$$\mathcal{E}_S = \{E \mid \omega_E = 0\}, \qquad \mathcal{E}_{S,2} = \{E \mid \omega_E = 0,\ \mathcal{L}_{\mathcal{C}} E = 2E\},$$

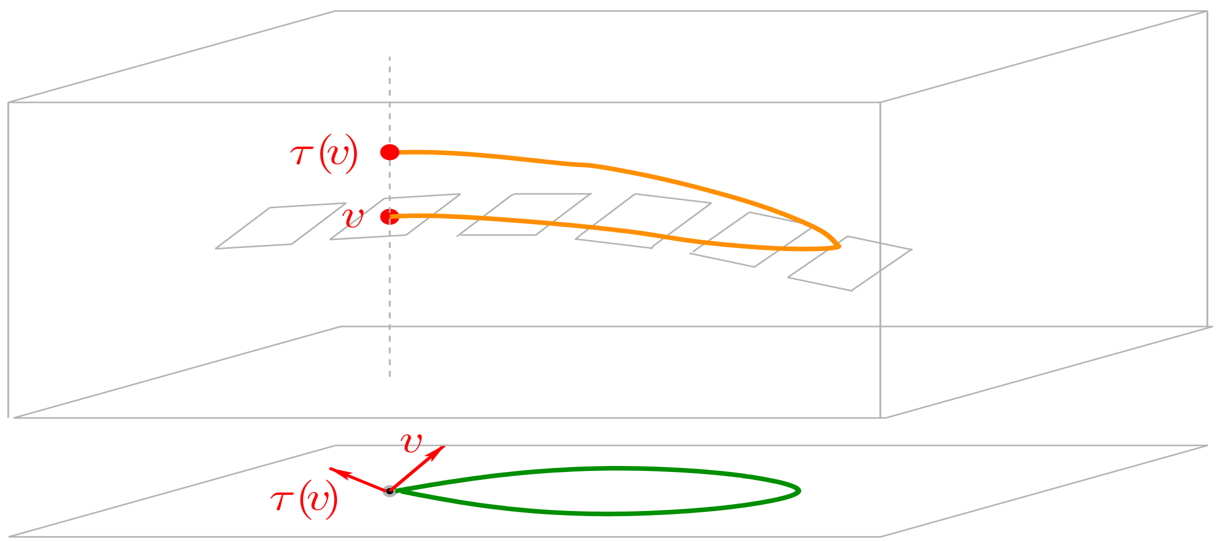
Parallel translation: geometric construction



- $R \equiv 0$



- $R \not\equiv 0$



Holonomy distribution

- $\mathcal{H} := \big\langle H\mathcal{T}M \big\rangle_{\mathcal{L}ie}$
- $\mathcal{H} = H\mathcal{T}M \oplus v\mathcal{H}, \qquad \operatorname{Im} R \subset v\mathcal{H},$
- $C^{\infty}_{hol} = \{E \ \mid \mathcal{L}_X E = 0,\ X \in \mathcal{H}\}\,, \qquad C^{\infty}_{hol,2} = \{E \in C^{\infty}_{hol} \mid \mathcal{L}_{\mathcal{C}} E = 2E\,\}\,,$

Proposition: $\mathcal{E}_{S,2} = C^{\infty}_{hol,2}$

Property: $\mathcal{E}_S$, C^{∞}_{hol}, $\mathcal{E}_{S,2} (= C^{\infty}_{hol,2})$ are vector space over $\mathbb{R}$.

Proposition: A 1-homogeneous functional combination of 2-homogeneous Euler-Lagrange functions is a 2-homogeneous Euler-Lagrange function.

$$E(x,y) := \varphi\big(E_1(x,y), \ldots, E_r(x,y)\big).$$

$$\mathcal{L}_X E = \frac{\partial \varphi}{\partial z^1} \cdot \mathcal{L}_X E_1 + \cdots + \frac{\partial \varphi}{\partial z^r} \cdot \mathcal{L}_X E_r = 0, \qquad \forall X \in \mathcal{H}$$

Definition: The *degree of metric freedom* m_S of a metrizable spray S is the maximal number of functionally independent elements of $\mathcal{E}_{S,2}$. If the spray S is non-metrizable, then we set $m_S = 0$.

Theorem: If S is metrizable and $\mathcal{H}$ is regular, then

$$\boxed{m_S = \operatorname{codim} \mathcal{H}.}$$

Remark: From the hypothesis of the Theorem one cannot omit the metrizability. There are examples for not metrizable sprays with $\operatorname{codim} \mathcal{H} > 0$.

Corollary: If S is isotropic, then

- $m_S = 0$ if and only if $R \neq 0$ and S is not metrizable;
- $m_S = 1$ if and only if $R \neq 0$ and S is metrizable;
- $m_S = n$ if and only if $R = 0$.

Explicite examples

- $\operatorname{codim}\mathcal{H} = 0$, $m_S = 0$: $\quad f^i := \sqrt{x^2 (y^1)^2 + (y^2)^2}\, y^i + (-1)^i \frac{y^1 y^i}{2x^2},$

- $\operatorname{codim}\mathcal{H} = 1$, $m_S = 1$: $\quad f^i = \dfrac{\mu\langle x, y\rangle}{1 + \mu|x|^2} y^i, \qquad \mu \in \mathbb{R} \setminus \{0\},$

- $\operatorname{codim}\mathcal{H} = n$, $m_S = n$: $\quad f^i = \dfrac{\langle a, y\rangle}{1 + \langle a, x\rangle} y^i,$

- $\operatorname{codim}\mathcal{H} > 0$, $m_S = 0$: $\quad f^1 = -\dfrac{(y^1)^2}{2x^2}, \qquad f^2 = 0.$

References

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Thank You